*The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane.**

Part First: Geometric Representation.†

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§1.—INTRODUCTION.

1. The group of Cremona transformations, whose geometric representation is the subject of the following paper, is a special case, $n=5$, of the general cross-ratio group of order $n!$.‡ If these cross-ratio transformations be expressed in homogeneous point coordinates, then each takes the form

$$z_1':z_2':z_3' = \phi(z_1, z_2, z_3):\chi(z_1, z_2, z_3):\psi(z_1, z_2, z_3).$$

2. The transformations of this system possess the following properties:

(a). They are all algebraic of order

$$\mu \leqq 2.$$

(b). They are birational, those of order $\mu = 2$ having 2^2-1 fundamental points at which the functions ϕ, χ, ψ vanish simultaneously.

(c). If S_i, S_j, S_k are the substitutions on the five indices with which the transformations A_i, A_j, A_k are respectively associated, and if

$$S_i S_j = S_k,$$

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* The study of this group was undertaken as a dissertation at the University of Chicago at the suggestion and under the direction of Professor Moore. A brief abstract of the chief results was printed in Science, July 29, 1898.

† In Part Second the complete form-system of invariants of the group is discussed.

‡ E. H. Moore, "The Cross-Ratio group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions" (American Journal of Mathematics, vol. XXII, pp. 336–342, 1900).

when compounded from left to right, then

$$A_i A_j = A_k,$$

when compounded from right to left.

3. This system of transformations forms a group distinguished by the following characteristics:

(a). It is holoedrically isomorphic with the symmetric substitution group on five indices.

(b). It is a group of *quadratic Cremona transformations of the plane into itself.*

(c). It has four fundamental points, some three of which belong to every quadratic transformation in the group. See Art. 13.

(d). It is abstractly identical with, and under collineation equivalent to, the Cremona group of order 120 noted by Autonne* and Kantor;† but it is distinguished by the fact that its operators are all *cross-ratio transformations.*‡

4. In studying the geometric representation of this group, the following generational transformations are chosen. Corresponding to the substitutions

(34), (23)(45), (45), (15)(34) and (12)

on the indices 1 .... 5, we find the transformations respectively,

$$\begin{aligned}
&K; \quad z_1' : z_2' : z_3' = z_3 : z_2 : z_1, \\
&L; \quad z_1' : z_2' : z_3' = z_3 - z_2 : z_3 - z_1 : z_3, \\
&M; \quad z_1' : z_2' : z_3' = z_2 : z_1 : z_3, \\
&T; \quad z_1' : z_2' : z_3' = z_3(z_1 - z_2) : (z_1 - z_2)(z_3 - z_2) : z_1(z_3 - z_2), \\
&T'; \quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2.
\end{aligned}$$

5. Of these, K, L and M, are collineations and T and T' are quadratic transformations. It is easily seen that all transformations corresponding to permutations on the indices 1 .... 5 which leave 1 fixed are linear, and that all others are quadratic. Hence the 24 linear transformations by themselves form a subgroup $G_{24}^{(1)}$, which, it will be found, may be generated by K, L, M (Art. 7), and then either T or T' will extend $G_{24}^{(1)}$ to the main group G_{120}.

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* Journal de Mathématiques, series 4, vol. I, 1885, p. 435.

† "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene," p. 105.

‡ E. H. Moore, American Journal of Mathematics, vol. XXII, p. 340, 1900.

§2.—*Geometric Representation of* $G_{24}^{(1)}$.

6. The Klein's* linear homogeneous substitution group $G_{4!}$ has been exhibited geometrically† in connection with the complete quadrangle (including the diagonals) whose vertices are†

$$\left.\begin{array}{ll} O_1; & x_1:x_2:x_3 = 1:1:1, \\ O_2; & x_1:x_2:x_3 = 1:-1:1, \\ O_3; & x_1:x_2:x_3 = 1:1:-1, \\ O_4; & x_1:x_2:x_3 = -1:1:1, \end{array}\right\} \tag{1}$$

where the boundaries of the fundamental region are defined by‡

$$\left.\begin{array}{r} x_3 \geqq 0, \\ x_1 - x_2 \leqq 0, \\ x_1 - x_3 \geqq 0. \end{array}\right\} \tag{2}$$

And the generators of the group are

$$\left.\begin{array}{ll} K'; & x_1':x_2':x_3' = x_3:x_2:x_1, \\ L'; & x_1':x_2':x_3' = x_1:x_2:-x_3, \\ M'; & x_1':x_2:x_3' = x_2:x_1:x_3. \end{array}\right\} \tag{3}$$

7. The linear subgroup $G_{24}^{(1)}$ possesses properties similar to those shown by Professor Moore for $G_{4!}$. As a substitution group, it permutes among themselves the four indices 2, 3, 4, 5.

Geometrically, it permutes among themselves certain four points of the plane which may be found by considering the transformations K, L, M set up in Art. 4.

K is a projective reflection on the axis

$$z_1 - z_3 = 0$$

from the center

$$z_1:z_2:z_3 = -1:0:1.$$

L and M are ordinary reflections on the axes respectively,

$$z_1 + z_2 - z_3 = 0,$$
$$z_1 - z_2 \qquad = 0.$$

* Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346–358, 1871).

† E. H. Moore, "Concerning Klein's Groups of $(n+1)!$ n-ary Collineations" (American Journal of Mathematics, vol. XXII, pp. 336–342, 1900).

‡ The fundamental region and the generators are here chosen in a way suitable for convenient use in this paper.

The plot of these lines [Fig. I]* shows that K, L, M permute the 4 points:

$$\left.\begin{array}{ll} Q_2; & z_1:z_2:z_3 = 1:1:1, \\ Q_3; & z_1:z_2:z_3 = 0:0:1, \\ Q_4; & z_1:z_2:z_3 = 1:0:0, \\ Q_5; & z_1:z_2:z_3 = 0:1:0. \end{array}\right\} \tag{4}$$

Now, a transformation can be found which throws

$$\begin{array}{cccc} O_1 & O_2 & O_3 & O_4 \end{array}$$

to

$$\begin{array}{cccc} Q_2 & Q_5 & Q_3 & Q_4 \end{array}$$

in the order indicated, namely,

$$\left.\begin{array}{lll} \text{direct:} & x_1:x_2:x_3 = & -z_1+z_2+z_3:z_1-z_2+z_3:z_1+z_2-z_3, \\ \text{inverse:} & z_1:z_2:z_3 = & x_2+x_3:x_3+x_1:x_1+x_2. \end{array}\right\} \tag{5}$$

8. Furthermore, this transformation also throws the generators

$$\begin{array}{ccc} K' & L' & M' \end{array}$$

to

$$\begin{array}{ccc} K & L & M. \end{array}$$

Hence, it follows at once that K, L, M are the *generators* of $G_{24}^{(1)}$, and that Q_2, Q_3, Q_4, Q_5 are permuted among themselves by *every* transformation of the linear subgroup.

Because of the projective relation thus established between $G_{4!}$ and $G_{24}^{(1)}$, the complete quadrangle [with its diagonal lines] whose vertices are Q_2, Q_3, Q_4, Q_5 is the geometric representation for the latter group. See Fig. II.

The fundamental region† for the new configuration is derived from that defined in (2) by the transformation (5). It is defined‡ by

$$\left.\begin{array}{r} z_1 - z_2 \geqq 0, \\ z_1 - z_3 \leqq 0, \\ z_1 + z_2 - z_3 \geqq 0. \end{array}\right\} \tag{6}$$

The new generators are, as they should be, edge-operators with reference to this region.

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*For convenience, Fig's. I, VII, VIII, IX are placed together on the last page of plates.

†An independent determination of the fundamental region (6) is given in Arts. 51 to 57, where the transformation names of all the regions are derived, and certain generational relations are discovered from the configuration itself.

‡ Where z_1, z_2, z_3 are all + above $z_2 = 0$ and to the right of $z_1 = 0$.

The plane is divided into 24 regions which are permuted among themselves by every transformation of the group. That is, if we call this configuration Π_1,

$$G_{24}^{(1)}\Pi_1 = \Pi_1. \tag{7}$$

§3.—*The Subgroups Conjugate with $G_{24}^{(1)}$.*

9. The general theory* of Cremona transformations may be illustrated for the quadratic operators of G_{120} by considering the product $MT \sim (1534)$† [Art. 4],

$$\left.\begin{array}{l} z_1' : z_2' : z_3' = (z_1 - z_2)(z_3 - z_2) : z_3\,(z_1 - z_2) : z_1\,(z_3 - z_2), \\ \text{of which the inverse is} \\ z_1 : z_2 : z_3 = z_3'\,(z_2' - z_1') : (z_2' - z_1')(z_3' - z_1') : z_2'\,(z_3' - z_1'). \end{array}\right\} \tag{1}$$

In the z-plane, the fundamental points are

$$\left.\begin{array}{ll} Q_2; & z_1 : z_2 : z_3 = 1 : 1 : 1, \\ Q_3; & z_1 : z_2 : z_3 = 0 : 0 : 1, \\ Q_4; & z_1 : z_2 : z_3 = 1 : 0 : 0, \end{array}\right\} \tag{2}$$

and the fundamental lines are

$$\left.\begin{array}{lr} Q_2\,Q_3; & z_1 - z_2 = 0, \\ Q_2\,Q_4; & z_2 - z_3 = 0, \\ Q_3\,Q_4; & z_2 = 0, \end{array}\right\} \tag{3}$$

while in the z'-plane the fundamental points are

$$\left.\begin{array}{ll} Q_2'; & z_1' : z_2' : z_3' = 1 : 1 : 1, \\ Q_3'; & z_1' : z_2' : z_3' = 0 : 0 : 1, \\ Q_5'; & z_1' : z_2' : z_3' = 0 : 1 : 0, \end{array}\right\} \tag{4}$$

and the fundamental lines are

$$\left.\begin{array}{lr} Q_2'\,Q_3'; & z_1' - z_2' = 0, \\ Q_2'\,Q_5'; & z_1' - z_3' = 0, \\ Q_3'\,Q_5'; & z_1' = 0, \end{array}\right\} \tag{5}$$

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*Clebsch, "Vorlesungen über Geometrie," vol I, pp. 474–496.

†Read, "The Operator MT Corresponding to the Substitution (1534)."

10. Evidently any operator whose square is identity, has the *same* fundamental points and lines in *both* planes (considered as superposed), though they are not necessarily associated in the same order in the two planes.

Indeed, this property is true of any one of the set of 6 transformations [Art. 19]

$$D_{1i}^{-1} \sim \{jkl\} \text{ all } (1i),^* \quad [i \neq j, \neq k, \neq l, = 2, 3 \ldots . 5],$$

In particular, any one of the set D_{12}^{-1} has the *coordinate vertices and sides* in some order as its fundamental points and lines in each plane.

Of these the quadratic inversion [Art. 4],

$$T' \sim (12)\,; \quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2$$

is the simplest, having always a vertex associated with its *opposite* side in the coordinate triangle.

11. Consider the transformation under MT from the z-plane to the z'-plane:

(a) A straight line (in general)

$$\alpha z_1 + \beta z_2 + \gamma z_3 = 0$$

goes into a *non-degenerate* conic

$$\alpha z_3' (z_2' - z_1') + \beta (z_2' - z_1')(z_3' - z_1') + \gamma z_2' (z_3' - z_1') = 0,$$

passing through the three points (4).

(b) A line through one of the points (2)

$$\alpha z_2 + \beta (z_1 - z_2) = 0$$

goes into a *degenerate* conic consisting of another line through *some one of the points* (4) *and its opposite fundamental side,*

$$z_1' - z_2' = 0,$$
$$\alpha z_3' - (\alpha - \beta) z_1' = 0.$$

In particular, a *side* of the quadrangle Π_1 through *one* fundamental point goes into a degenerate conic consisting of *two sides,* thus:

$$z_1 - z_3 = 0$$

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* I use the notation of Cayley, Quarterly Journal, vol. 25, p. 73, except that I use the parentheses to indicate substitutions, and hence, indicate groups by $\{\ \}$, so that the above form means the group $\{jkl\}$ all multiplied by the substitution $(1i)$ while $\{jkl\}$ all $\{1i\}$ means the same group multiplied by the group 1, $(1i)$.

goes into $z_1' = 0$ and $z_2' - z_3' = 0$.

(c) A *fundamental* line, joining *two* of the points (2),

$$z_1 - z_2 = 0$$

goes into a degenerate conic consisting of two of the fundamental lines (5), namely

$$z_1' = 0$$

and

$$z_1' - z_2' = 0.$$

See also (f) below for the correspondence of the points on such a line.

(d) An *ordinary* point, defined by the intersection of two (general) lines, goes into the fourth (movable) point of intersection of the two corresponding conics, the other three intersections being the fundamental points in the z'-plane.

(e) One of the fundamental points (3) corresponds to one of the fundamental sides (5) in the following sense: to each "*direction*" in which a movable point may approach the given fundamental point in the z-plane, there corresponds a definite point situated on a certain one of the fundamental lines in the z'-plane. Any such "direction" is determined by a *specific tangent in the fundamental point* and the corresponding point in the z'-plane is given by (b) above. For example, to the direction given by the tangent

$$\alpha z_2 + \beta (z_1 - z_2) = 0 \tag{6}$$

in the fundamental point

$$z_1 : z_2 : z_3 = 0 : 0 : 1$$

there corresponds the point in the z'-plane given by the intersection of the line

$$\alpha z_3' - (\alpha - \beta) z_1' = 0$$

with the fundamental side

$$z_1' - z_2' = 0. \tag{7}$$

Thus, when the tangent (6) varies through all directions, the fundamental side (7) becomes the locus of the corresponding points.

(f) *A fundamental point must, therefore, be regarded as a "pencil of directions,"* and this makes clear how a fundamental line in the z-plane goes, by (c), into two fundamental lines in the z'-plane. For to each point on such a line, there corresponds, by (e), a specific direction in a certain pencil of the z'-plane. Among these points, however, are two fundamental points, to each of which corresponds,

by the infinity of directions through it, a certain fundamental line in the z'-plane. These lines are themselves, therefore, two direction-tangents in the above pencil, which is, then, determined by their intersection.

12. Since, under the transformation MT, the pencils of directions (2) go, in some order, into the ranges of points (5), the former are called *the critical points* and the latter *the critical lines of the transformation.* All other points are non-critical under the transformation MT.

Likewise (4) and (3) are the critical points and lines of the inverse transformation $(MT)^{-1}$. Evidently $(MT)^{-1}$ throws any line of the plane into a conic passing through the critical points of MT.

13. Since, for every quadratic transformation in G_{120}, the critical points are certain three (n^2-1) of the four vertices, and the critical lines are three of the six sides of the complete quadrangle Π_1, previously found [Art. 8], the group is said to have four critical points and six critical lines.

Classification of the Quadratic Transformations, Arts. 14–16.

14. It has been seen that all linear transformations in G_{120} are associated with the substitutions which leave the index 1 fixed. The quadratic transformations may be divided into four sets of 24 each, according as the index 1 is thrown to 2, 3, 4 or 5 by the corresponding substitutions.

15. Lemma. *If* s_{1i} *is a transformation corresponding to any particular substitution throwing* 1 *to* i, *and if* S_{1i} *represents the whole set of such transformations, then*

$$S_{1i} = G_{24}^{(1)} s_{1i} \sim \{2345\} \text{ all } (1i), \qquad [i = 2, 3, 4, 5].$$

For the set S_{1i} must be such that when the corresponding substitutions are combined with the one belonging to s_{1i}^{-1}, there will result all substitutions leaving 1 fixed; that is, the substitution group corresponding to $G_{24}^{(1)}$, thus:

$$S_{1i}\, s_{1i}^{-1} = G_{24}^{(1)}.$$

Hence, multiplying on the right by s_{1i},

$$S_{1i} = G_{24}^{(1)} s_{1i} \tag{8}$$

or

$$S_{1i}^{-1} = (G_{24}^{(1)} s_{1i})^{-1} = s_{1i}^{-1}\, G_{24}^{(1)}. \tag{9}$$

16. THEOREM. *The critical points are the same for all quadratic transformations belonging to the same set* S_{1i}, $(i = 2, 3, 4, 5)$.

For if s_{1i}^{-1}, any particular transformation of the set S_{1i}^{-1}, be applied to the complete quadrangle Π_1 [Art. 8] a definite configuration will result in which every line of Π_1 will have become a conic passing through three fixed points, namely, the critical points of s_{1i} [Art. 12]. If the new figure be called Π_i, $[i = 2, 3, 4, 5]$, then

$$s_{1i}^{-1}\Pi_1 = \Pi_i. \tag{10}$$

Any other transformation of the set S_{1i}^{-1}, when applied to Π_1, will produce the same figure Π_i, for, using equation (9),

$$S_{1i}^{-1}\Pi_1 = s_{1i}^{-1}\, G_{24}^{(1)}\,\Pi_1.$$

By equations (7) of Art. 8 and (10) above,

$$s_{1i}^{-1}\, G_{24}^{(1)}\,\Pi_1 = s_{1i}^{-1}\Pi_1 = \Pi_i.$$

Hence,

$$S_{1i}^{-1}\Pi_1 = \Pi_i. \tag{11}$$

Since each transformation of the set S_{1i}^{-1} throws the lines of Π_1 into conics intersecting in the same fixed points in Π_i, therefore, all the transformations in the set S_{1i} have the same critical points [Art. 12].

Configurations for the Subgroups $G_{24}^{(i)}$, Arts. 17–24.

17. If $G_{24}^{(1)}$ be transformed by any one of the set of transformations S_{1i}, the new subgroup is the same as that produced by any other transformer of the set, for if

$$s_{1i}^{-1}\, G_{24}^{(1)}\, s_{1i} = G_{24}^{(i)},$$

then, by use of (8) and (9),

$$S_{1i}^{-1}\, G_{24}^{(1)}\, S_{1i} = G_{24}^{(i)}. \tag{12}$$

This $G_{24}^{(i)}$ corresponds to the substitution group which leaves the index i fixed and permutes among themselves $1, j, k, l$, $[i \neq j, \neq k, \neq l, = 2, 3, 4, 5]$.

To $G_{24}^{(i)}$ belongs a configuration related to Π_1 in the following manner:

Let g and g' be any pair of corresponding transformations in $G_{24}^{(1)}$ and $G_{24}^{(i)}$ respectively, so that

$$s_{1i}^{-1}\, g\, s_{1i} = g'.$$

Suppose P to be any point of the plane and P' its conjugate under g', then, by (13),

$$s_{1i}^{-1}\, g\, s_{1i}\, P = P'. \tag{14}$$

If, now, $s_{1i}\, P = P_1$ and $g P_1 = P_2$, then must

$$s_{1i}^{-1}\, P_2 = P'.$$

Thus P_1 and P_2, a pair of conjugate points under g, are thrown by s_{1i}^{-1} to P and P' respectively, a pair of conjugate points under g'.

Since (13) holds for every g and for every s_{1i}^{-1}, therefore, any transformation of the set S_{1i}^{-1}, operating on a pair of conjugate points under $G_{24}^{(1)}$, gives a corresponding pair of conjugate points under $G_{24}^{(i)}$. Therefore, the configuration connected with $G_{24}^{(i)}$ is derived by applying to Π_1 any transformation of the set S_{1i}^{-1} and the result is by (11) Π_i, $\quad (i = 2 \ldots . 5)$.

18. THEOREM. *The configuration Π_i is thrown into itself by all transformations of $G_{24}^{(i)}$.*

For by (12), $\qquad G_{24}^{(i)}\,\Pi_i = S_{1i}^{-1}\, G_{24}^{(1)}\, S_{1i}\,\Pi_i$

and by (11), $\qquad \Pi_i = S_{1i}^{-1}\,\Pi_1.$

Hence, $\qquad S_{1i}\,\Pi_i = S_{1i}\, S_{1i}^{-1}\,\Pi_1 = \Pi_1,$

and, Art. 8, $\qquad G_{24}^{(1)}\, S_{1i}\,\Pi_i = G_{24}^{(1)}\,\Pi_1 \quad = \Pi_1.$

Then, by (11), $\qquad S_{1i}^{-1}\, G_{24}^{(1)}\, S_{1i}\,\Pi_i = S_{1i}^{-1}\,\Pi_1 \quad = \Pi_i.$

Therefore, $\qquad G_{24}^{(i)}\,\Pi_i = \Pi_i. \qquad (15)$

19. The notation for the 4 points in Π_1 has been so chosen [Art. 7] that the three critical points for the set of transformers S_{1i} are

$$Q_j,\ Q_k,\ Q_l. \qquad [i \neq j, \neq k, \neq l, = 2, 3, 4, 5].$$

These are the three points through which pass all conics in Π_i.

Thus Q_i plays a particular rôle in the passage from Π_1 to Π_i, in that by any of the six transformations in the set D_{1i}^{-1} [Art. 10], it is unmoved, and the other three critical points become ranges of points on

$$Q_i\,Q_j,\ \ Q_i\,Q_k,\ \ Q_i\,Q_l,$$

while, by any other* transformation of the system S_{1i}^{-1}, some one of the three is carried to Q_i and the other two with Q_i become ranges of points on

$$Q_i\,Q_j,\quad Q_i\,Q_k,\quad Q_i\,Q_l.$$

20. It follows that in operating upon Π_1 by any one of the set of transformations

$$S_{1i}^{-1}, \qquad (i = 2,\ 3,\ 4,\ 5),$$

the resulting configuration, Π_i, contains again the 4 *pencils* and 6 *sides* of Π_1, while the diagonal lines become proper conics in Π_i. It is convenient for this purpose to choose the following special transformations:

$$\begin{aligned}
s_{12} &\sim (12) \quad ; \quad z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2,\\
s_{13} &\sim (132) \quad ; \quad z_1' : z_2' : z_3' = z_3(z_3 - z_2) : z_3(z_3 - z_1) : (z_3 - z_1)(z_3 - z_2),\\
s_{14} &\sim (1452) \quad ; \quad z_1' : z_2' : z_3' = z_1(z_1 - z_3) : (z_1 - z_3)(z_1 - z_2) : z_1(z_1 - z_2),\\
s_{15} &\sim (152) \quad ; \quad z_1' : z_2' : z_3' = z_2(z_2 - z_3) : (z_2 - z_3)(z_2 - z_1) : z_2(z_2 - z_1).
\end{aligned}$$

The generators of the subgroups $G_{24}^{(i)}$ are found by transforming K, L, M through s_{12}, s_{13}, s_{14}, s_{15} respectively. The three conics of Π_i are given by operating upon the diagonal lines of Π_1 by s_{12}^{-1}, s_{13}^{-1}, s_{14}^{-1}, s_{15}^{-1} respectively. The boundaries of the fundamental regions of Π_i are shown by operating upon those of Π_1 by s_{12}^{-1}, s_{13}^{-1}, s_{14}^{-1}, s_{15}^{-1} respectively.

The configurations, $\Pi_2 \ldots\ldots \Pi_5$, are shown in the figures III, IV, V, VI.

§4.—The Configuration for G_{120}.

Algebraic Study of the Configuration, Arts. 21–27.

21. If, now, the figures Π_i be superposed upon the complete quadrangle Π_1, the resulting configuration Π admits the following algebraic verification as to the intersections of the twelve conics with the sides and diagonals of Π_1.

22. Since [Art. 19] the three conics of Π_i each pass through

$$Q_j,\ Q_k,\ Q_l, \qquad [i \neq j, \neq k, \neq l, = 2, 3, 4, 5],$$

but none of them through the fourth vertex, therefore, through each vertex pass

$$3 \,.\, 3 = 9 \text{ conics}.$$

23. Since [Art. 20] the 6 sides and 4 pencils of Π_1 are reproduced in Π_i, while the 3 diagonals in each case become proper conics, the intersections of Π

* The set D_{1i}^{-1} is included in the set S_{1i}^{-1}. See Art 15.

will evidently be of three kinds only, namely, (a) *conics with the sides*, (b) *conics with the diagonals*, (c) *conics with conics*.

24. Each side, in addition to the 2 vertices lying on it, through each of which pass 9 conics, has 2 other rational points through each of which passes one conic, thus:

On the sides	lie the rational points.
$z_1 = 0$	$0 : 1 : 2$ and $0:2:1$
$z_1 - z_3 = 0$	$1 : -1 : 1$ and $2:1:2$
$z_2 = 0$	$1 : 0 : 2$ and $2:0:1$
$z_2 - z_3 = 0$	$-1 : 1 : 1$ and $1:2:2$
$z_1 - z_2 = 0$	$1 : 1 : -1$ and $2:2:1$
$z_3 = 0$	$1 : 2 : 0$ and $2:1:0$

25. Each side has also one intersection with a diagonal line. It will be found in the succeeding grouptheoretic study that these belong to the same class as the intersections with the conics.

26. Each diagonal line has 4 points, through each of which pass 4 conics, one belonging to each of the figures Π_i.

Thus, on the lines

$$z_1 + z_2 - z_3 = 0, \quad z_1 - z_2 - z_3 = 0, \quad z_1 - z_2 + z_3 = 0$$

lie respectively the systems of 4 points:

$$\begin{array}{lll}
-\lambda : 1+\lambda : 1 & \lambda : -\lambda' : 1 & -\lambda : \lambda' : 1 \\
-\lambda' : 1+\lambda' : 1 & \lambda' : -\lambda : 1 & -\lambda' : \lambda : 1 \\
1+\lambda : -\lambda : 1 & 1+\lambda : \lambda : 1 & \lambda : 1+\lambda : 1 \\
1+\lambda' : \lambda' : 1 & 1+\lambda' : \lambda' : 1 & \lambda' : 1+\lambda' : 1
\end{array}$$

wherein

$$\lambda = \frac{1+\sqrt{5}}{2}, \quad \lambda' = \frac{1-\sqrt{5}}{2}.$$

27. The only real intersections of the 12 conics among themselves are at the 4 vertices and at the above 12 irrational points on the 3 diagonals. But there is a set of 20 imaginary points, through each of which pass 3 conics. These will be discussed in the succeeding grouptheoretic study. The configuration Π is shown in Fig. X.

Grouptheoretic Study of the Configuration Π, Arts. 28–46.

28. In order to discuss the systems of conjugate elements under the main group, a generating operator is needed which extends $G_{24}^{(1)}$ to G_{120}. *Any* transformation of G_{120} not contained in $G_{24}^{(1)}$ will serve, since the index of $G_{24}^{(1)}$ under G_{120} is a prime.

It will be convenient to choose for this purpose

$$T \sim (15)(34), \qquad \text{[see Art. 4]},$$

since K, L and T are edge-operators for the region defined by

$$\begin{aligned} z_1 - z_3 &\leqq 0, \\ z_1 + z_2 - z_3 &\geqq 0, \\ z_1 z_2 - z_1 z_3 + z_2 z_3 &\leqq 0, \end{aligned} \qquad \text{[see Fig. XI]}.$$

The old generator M is now expressible in terms of K, L, T, thus:

$$M = TKLKTLTK,$$

so that K, L, T generate the group G_{120}.

29. Under G_{120} any element—point, line or conic—will, in general, go into 120 conjugates. Certain elements may, however, go into fewer conjugates. Such a *special* element E is invariant under a subgroup $G\{E\}$, whose index under G_{120} indicates the number of elements in the conjugate system.

The element E may be designated by a notation n consisting of such a combination of the indices 1 .... 5 as will characterize the corresponding substitution group $G'\{E\}$. This notation may be indicated on the group by $G^N\{E\}$.

Let S be any transformation in G_{120}. Then, if E is invariant under $G\{E\}$, $S^{-1}E$ will be invariant under $S^{-1}G\{E\}S$.

Let s be the substitution corresponding to S. Then, if n is the notation for E, ns will be the notation for $S^{-1}E$, wherein s acts as a substitution upon the indices in the cycles of the notation n. Likewise, the notation for SE is ns^{-1}.

The complete set of substitutions changing the notation of E to that of SE is given by $G'\{E\}s^{-1}$, and the transformations corresponding to the inverse of the substitutions in the set are the only ones in G_{120} which throw E to SE.

30. As an illustration of such a special element, the diagonal

$$z_1 + z_2 - z_3 = 0,$$

which is fixed by points under the transformation corresponding to the substitution (23)(45), is found to be invariant under the subgroup

$$G_8^{23.45} \sim \{2435\}_8 .*$$

Its notation will then be 23 . 45, provided we agree

$$23.45 = 32.45 = 32.54 = 45.32 = 54.32 = 54.23, \text{ etc.};$$

that is, *the pairing is definitive, but not the sequence either of the pairs or the indices in a pair.*

The transformation $S \sim (134)$

throws $z_1 + z_2 - z_3 = 0$

to the conic $z_1 z_2 - 2z_2 z_3 + z_3^2 = 0.$

This conic is fixed by points under the transformation $\sim (15)(24)$ and is invariant under the subgroup

$$G_8^{15.24} = S^{-1} G_8^{23.45} S \sim (134)^{-1} \{2435\}_8 (134).$$

Its notation is $15.24 = [23.45](134).$†

The complete set of transformations throwing the diagonal 23 . 45 to the conic 15 . 24 is given by the substitutions

$$[\{2435\}_8 (134)]^{-1}.$$

31. Since the index of $G_8^{23.45}$ under G_{120} is 15, it would at once appear that the 3 diagonals and 12 conics form a conjugate system. This is shown in the following table,‡ in which the generators K, L and T are applied successively to the diagonal, while its notation is transformed by the *inverse* of the correspond-

* Notation of Cayley. See foot-note to Art. 10.

† Article 29. This means operate on the cycles of the *notation* 23 . 45 with the *substitution* (134).

‡ The result of operating on any diagonal or conic of the configuration by any transformation of G_{120} may be read at once from this table without algebraic computation.

ing substitutions (these being the same as the *direct*, since K, L, T are all of period 2).

Diagonals and Conics.	Notation.	$K \sim (34)$.	$L \sim (23)(45)$.	$T \sim (15)(34)$.
$z_1 + z_2 - z_3 = 0$	23 . 45	24 . 35	23 . 45	13 . 24
$z_1 - z_2 - z_3 = 0$	24 . 35	23 . 45	24 . 35	14 . 23
$z_1 - z_2 + z_3 = 0$	25 . 34	25 . 34	25 . 34	12 . 34
$z_2^2 - z_1 z_3 = 0$	12 . 34	12 . 34	13 . 25	25 . 34
$z_1^2 - z_2 z_3 = 0$	12 . 35	12 . 45	13 . 24	14 . 25
$z_1 z_2 - z_3^2 = 0$	12 . 45	12 . 35	13 . 45	13 . 25
$z_2^2 - 2z_2 z_3 + z_1 z_3 = 0$	13 . 24	14 . 23	12 . 35	23 . 45
$z_1^2 - 2z_1 z_3 + z_2 z_3 = 0$	13 . 25	14 . 25	12 . 34	12 . 45
$z_1 z_2 - z_1 z_3 - z_2 z_3 = 0$	13 . 45	14 . 35	12 . 45	13 . 45
$z_2^2 - 2z_1 z_2 + z_1 z_3 = 0$	14 . 23	13 . 24	15 . 23	24 . 35
$z_1 z_2 - 2z_1 z_3 + z_3^2 = 0$	14 . 25	13 . 25	15 . 34	12 . 35
$z_1 z_2 + z_1 z_3 - z_2 z_3 = 0$	14 . 35	13 . 45	15 . 24	14 . 35
$z_1^2 - 2z_1 z_2 + z_2 z_3 = 0$	15 . 23	15 . 24	14 . 23	15 . 24
$z_1 z_2 - 2z_2 z_3 + z_3^2 = 0$	15 . 24	15 . 23	14 . 35	15 . 23
$z_1 z_2 - z_1 z_3 + z_2 z_3 = 0$	15 . 34	15 . 34	14 . 25	15 . 34

The 3 diagonals

$$2i\,.\,jk \qquad (i, j, k, = 3, 4, 5)$$

belong to the *linear* subgroups

$$G_8^{2i\,.\,jk} \sim \{2jik\}_8.$$

The 12 conics $1i\,.\,jk$ $(i, j, k, = 2, 3, 4, 5)$
belong the *quadratic* subgroups

$$G_8^{1i\,.\,jk} \sim \{1jik\}_8.$$

32. Again, the side $z_1 = 0$

is found to be *fixed by points* under the transformation $\sim(24)$ and invariant under the subgroup

$$G_{12}^{24} \sim \{135\} \text{ all } \{24\}.^*$$

Its notation will then be $24 = 42$.

Since the index of G_{12}^{24} is 10, *the* 6 *sides do not form a complete system*, but we have seen that under quadratic transformations a *pencil of directions* plays the same rôle as a *side of the quadrangle* Π_1, so that the 4 pencils and 6 sides may form a conjugate system. In fact the transformation

$$S \sim (12)(34)$$

throws the side $z_1 = 0$

to the pencil at $z_1 : z_2 : z_3 = 0 : 0 : 1$.

This pencil is fixed by *directions* under the transformation $\sim(13)$ and is invariant under the subgroup

$$G_{12}^{13} = S^{-1} G_{12}^{24} S \sim [(12)(34)]^{-1}[\{135\} \text{ all } \{24\}][(12)(34)].$$

The notation for the pencil is

$$13 = 24\,[(12)(34)].\dagger$$

The complete set of operators throwing the side 24 into the pencil 13 is given by the substitutions

$$[\{135\} \text{ all } \{24\}][(12)(34)].$$

33. The notation for the 10 elements and the proof that they form a closed system under the generators K, L, T, is as follows:

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* See foot-note to Art. 10.

† This means operate upon the *notation* 24 with the *substitution* (12)(34).

Sides and Pencils.	Notation.	$K \sim (34)$.	$L \sim (23)(45)$.	$T \sim (15)(34)$.
$1:1:1$	12	12	13	25
$0:0:1$	13	14	12	45
$1:0:0$	14	13	15	35
$0:1:0$	15	15	14	15
$z_3 = 0$	23	24	23	24
$z_1 = 0$	24	23	35	23
$z_2 = 0$	25	25	34	12
$z_1 - z_3 = 0$	34	34	25	34
$z_2 - z_3 = 0$	35	45	24	14
$z_1 - z_2 = 0$	45	35	45	13

The 6 sides,

$$ij = ji, \qquad (i \neq j, = 2 \ldots . 5),$$

belong to the subgroups

$$G_{12}^{ij} \sim \{1kl\} \text{ all } \{ij\}, \qquad (k, l \neq i, j, = 2 \ldots . 5).$$

The 4 pencils

$$1i = i1 \qquad (i = 2 \ldots . 5),$$

belong to the subgroups

$$G_{12}^{1i} \sim \{jkl\} \text{ all } \{1i\}, \qquad (j, k, l, \neq i, = 2 \ldots . 5).$$

34. Each line or conic in the above system is a *locus of fixed points* under the transformation corresponding to the substitution by which it is named. These are the *only such loci* under the operators of G_{120}, as will at once appear by the following complete list of fixed points for the different types of transformations:

35. *Period* 2. *Two Types.*

(1). *Type* (ij).

(a). When $i \neq j, = 2 \ldots 5$, the side ij is a *locus of fixed points.*

One point not in the locus ij is fixed, namely, the intersection of the diagonal $ij \,.\, kl$ with the side kl. $(k, l, \neq i, j, = 2 \ldots 5)$.

One direction is fixed in each of the pencils, $1k$ and $1l$, namely, that one whose tangent passes through the fixed point outside the locus ij.

(b). When $i = 2 \ldots 5$, the pencil $1i$ is a *centre* of *fixed directions* under the transformations $s_{1i} \sim (1i)$.

One direction is also fixed under these transformations in each of the pencils

$$1j,\ 1k,\ 1l, \qquad (j, k, l, \neq i, = 2 \ldots 5).$$

And one point is fixed in each of the sides passing through the vertex $1i$, namely,

$$jk,\ jl,\ kl.$$

(2). *Type* $(ij)(kl)$.

(a). When $i, j, k, l, = 2 \ldots 5$, the diagonal $ij \,.\, kl$ is a *locus* of *fixed points* under the transformation $\sim (ij)(kl)$ and one point not in $ij \,.\, kl$ is fixed under the same transformation, namely, the intersection of the other two diagonals.

(b). When $i = 1$ and $j, k, l = 2 \ldots 5$, the conic $1j \,.\, kl$ is a *locus* of *fixed points* under the transformation $\sim (1j)(kl)$, and one direction also is fixed in the pencil $1i$, that one whose tangent is common to the conics

$$1k \,.\, jl \text{ and } 1l \,.\, jk.$$

36. *Period* 3. *One type,* (ijk). $(i, j, k = 1 \ldots 5)$.

There is no locus of fixed points for this type, but there are two classes of *discrete fixed* points, namely,

(a). A pair of imaginary points through which pass the three conics [Art. 42]

$$ij \,.\, lm,\ \ ik \,.\, lm,\ \ jk \,.\, lm, \qquad (i, j, k, l, m, = 1 \ldots 5).$$

(b). A pair of imaginary points lying on the side (or imaginary directions in the pencil) (Art. 46)

$$lm, \qquad (l, m = 1 \ldots 5).$$

37. *Period* 4. *One type*, $(ijkl)$.

There is no locus of fixed points, but again two classes of *discrete* fixed points.

(a). Real points where two sides meet two diagonals. Such a *point* on a *side* is conjugate with a *direction* in a *pencil* whose tangent (one of the sides) is common to the two conics corresponding to the two diagonals. See Art. 43.

(b). *Imaginary points* which lie in pairs on the diagonals or conics. See Art. 45.

38. *Period* 5. *One type*, $(ijklm)$.

The only fixed points are real, discrete points whose coördinates involve the surd $\sqrt{5}$. These will be discussed in Art. 44.

39. *Period* 6. *One type*, $(ij)(klm)$.

Again, the only fixed points are imaginary, discrete points of the same form as class (b) under Period 3. See Art. 46.

40. We have thus found no loci of fixed points (or directions) except the 10 sides (and pencils) and 15 diagonals and conics which are exactly all the curves in the configuration Π.

In order now to discover all the systems of conjugate points, with their respective notations, we take any known fixed point under a given type of transformation and find the subgroup which leaves it unmoved and determine its notation by means of the corresponding substitution group. Then we operate upon it with the generators of G_{120}, transforming the notation as in Arts. 31, 33, and continue the process till the system is closed.

The intersection points of Π are classified as double, triple, quadruple and quintuple, according to the number of lines or conics passing through them.

Corresponding to a *double point lying on one of the sides will be a direction in one of the pencils, whose tangent belongs to one conic only, while to a quadruple point on one of the sides corresponds a direction in a pencil, whose tangent is common to two conics and coincides with a side passing through the pencil.*

Following is a complete enumeration of intersection points and their conjugate systems:

41. *Double Points.*

The point $$-1:0:1$$
is invariant under transformations of Period 2 corresponding to
$$(34),\ (25) \text{ and } (25)(34),$$
and under no others except identity. It, therefore, belongs to the subgroup
$$G_4^{25.34} \sim \{(25)(34)\}.$$
The point $$1:2:1$$
is also invariant under the same subgroup. The notations for these two points are, then,
$$\overline{25}\,.\,34 \text{ and } \overline{34}\,.\,25,$$
where the *sequence of pairs is definitive but not the sequence of indices in a pair.* The first pair indicates the name of the side which is cut at the point by the diagonal 25 . 34. The stroke distinguishes the notation from that of the diagonal 25 . 34 = 34 . 25. [Art. 30.]

The complete list, consisting of 12 directions in the 4 pencils plus 18 points on the 6 sides is as follows:

"Direction" Tangents in the Pencils.		Notation.
$z_1 + z_3 = 0$	15	$\overline{12}\,.\,34$
$2z_1 - z_3 = 0$	15	$\overline{14}\,.\,23$
$z_1 - 2z_3 = 0$	15	$\overline{13}\,.\,24$
$z_2 + z_3 = 0$	14	$\overline{12}\,.\,35$
$2z_2 - z_3 = 0$	14	$\overline{15}\,.\,23$
$z_2 - 2z_3 = 0$	14	$\overline{13}\,.\,25$
$z_1 + z_2 = 0$	13	$\overline{12}\,.\,45$
$z_1 - 2z_2 = 0$	13	$\overline{15}\,.\,24$
$2z_1 - z_2 = 0$	13	$\overline{14}\,.\,25$
$z_1 + z_2 - 2z_3 = 0$	12	$\overline{13}\,.\,45$
$z_1 - 2z_2 + z_3 = 0$	12	$\overline{15}\,.\,34$
$-2z_1 + z_2 + z_3 = 0$	12	$\overline{14}\,.\,35$

Intersections on the	Sides.	Notation.
$1 : 2 : 0$	23	$\overline{23}.14$
$2 : 1 : 0$	23	$\overline{23}.15$
$1 : -1 : 0$	23	$\overline{23}.45$
$0 : 2 : 1$	24	$\overline{24}.13$
$0 : 1 : 2$	24	$\overline{24}.15$
$0 : -1 : 1$	24	$\overline{24}.35$
$2 : 0 : 1$	25	$\overline{25}.13$
$1 : 0 : 2$	25	$\overline{25}.14$
$-1 : 0 : 1$	25	$\overline{25}.34$
$1 : -1 : 1$	34	$\overline{34}.12$
$2 : 1 : 2$	34	$\overline{34}.15$
$1 : 2 : 1$	34	$\overline{34}.25$
$-1 : 1 : 1$	35	$\overline{35}.12$
$1 : 2 : 2$	35	$\overline{35}.14$
$2 : 1 : 1$	35	$\overline{35}.24$
$1 : 1 : -1$	45	$\overline{45}.12$
$2 : 2 : 1$	45	$\overline{45}.13$
$1 : 1 : 2$	45	$\overline{45}.23$

These points or directions, which lie by threes on the 6 sides and 4 pencils, are fixed by pairs under the 15 *distinct* conjugate subgroups of order 4. That is, the points

$$\overline{ij}.kl \text{ and } \overline{kl}.ij$$

belong to the groups $G_4^{ij.kl} \sim \{(ij)(kl)\}$. $[i, j, k, l, = 1 \dots 5]$.

42. *Triple points.*

The two imaginary points*

$$1-\omega : 1-\omega^2 : 1 \text{ and } 1-\omega^2 : 1-\omega : 1$$

are fixed under the transformation $\sim (254)$, and are invariant under the sub-

* ω and ω^2 are imaginary cube roots of unity.

group

$$G_6^{13.254} \sim [\{254\} \text{ all } \{12\}] \text{ pos.}$$

These points may be named respectively

$$13.245 \text{ and } 13.254,$$

where $$13.245 = 31.254 \neq 31.245$$

and $$13.254 = 31.245 \neq 31.254.$$

That is, the meaning of the notation is unchanged by reversing the cyclic order of indices in *both parts simultaneously*, but *not in either part alone.*

The points of this system are as follows:

20 Triple Points.	Notation.
$\omega : \omega^2 : 1$	12 . 345
$\omega^2 : \omega : 1$	12 . 354
$1-\omega : 1-\omega^2 : 1$	13 . 245
$1-\omega^2 : 1-\omega : 1$	13 . 254
$1-\omega^2 : -3\omega^2 : 3$	14 . 235
$1-\omega : -3\omega : 3$	14 . 253
$-3\omega^2 : 1-\omega^2 : 3$	15 . 234
$-3\omega : 1-\omega : 3$	15 . 243
$1-\omega^2 : 1-\omega : 3$	23 . 145
$1-\omega : 1-\omega^2 : 3$	23 . 154
$1-\omega : -\omega : 1$	24 . 135
$1-\omega^2 : -\omega^2 : 1$	24 . 153
$-\omega : 1-\omega : 1$	25 . 134
$-\omega^2 : 1-\omega^2 : 1$	25 . 143
$\omega : -\omega^2 : 1$	34 . 125
$\omega^2 : -\omega : 1$	34 . 152
$-\omega^2 : \omega : 1$	35 . 124
$-\omega : \omega^2 : 1$	35 . 142
$-\omega : -\omega^2 : 1$	45 . 123
$-\omega^2 : -\omega : 1$	45 . 132

These points belong in pairs to the 10 distinct conjugate subgroups

$$G_6^{ij.klm} = [\{klm\} \text{ all } \{ij\}] \text{ pos.}$$

Namely, to $G_6^{ij.klm}$ belong the 2 points

$$ij \,.\, klm \text{ and } ij \,.\, kml, \qquad [i, j, k, l, m, = 1 \,\ldots\, 5].$$

43. *Quadruple points.*

The point $\quad 0:1:1$

is fixed under the transformations corresponding to

$$(24)(35) \text{ and } (2345),$$

and is invariant under the subgroup

$$G_8^{24.35} \sim \{2345\}_8.$$

And the direction in the pencil 15,

$$z_1 - z_3 = 0,$$

is fixed under the transformations

$$(15)(34) \text{ and } (1354),$$

and is invariant under the subgroup

$$G_8^{15.34} \sim \{1354\}_8.$$

We name this point and direction

$$\overline{24} \,.\, \overline{35} \text{ and } \overline{15} \,.\, \overline{34}$$

respectively, where the *pairing only* is definitive, the strokes serving to distinguish these from the conics 15 . 34, 24 . 35, and also from the double points

$$\overline{15} \,.\, 34, \quad \overline{34} \,.\, 15, \quad \overline{24} \,.\, 35, \quad \overline{35} \,.\, 24.$$

The complete set is as follows:

"Direction" Tangents in the Pencils.		Notation.
$z_1 - z_3 = 0$	12	$\overline{12}\,.\,\overline{34}$
$z_2 - z_3 = 0$	12	$\overline{12}\,.\,\overline{35}$
$z_1 - z_2 = 0$	12	$\overline{12}\,.\,\overline{45}$
$z_1 = 0$	13	$\overline{13}\,.\,\overline{24}$
$z_2 = 0$	13	$\overline{13}\,.\,\overline{25}$
$z_1 - z_2 = 0$	13	$\overline{13}\,.\,\overline{45}$
$z_3 = 0$	14	$\overline{14}\,.\,\overline{23}$
$z_2 = 0$	14	$\overline{14}\,.\,\overline{25}$
$z_2 - z_3 = 0$	14	$\overline{14}\,.\,\overline{35}$
$z_3 = 0$	15	$\overline{15}\,.\,\overline{23}$
$z_1 = 0$	15	$\overline{15}\,.\,\overline{24}$
$z_1 - z_3 = 0$	15	$\overline{15}\,.\,\overline{34}$
$0:1:1$		$\overline{24}\,.\,\overline{35}$
$1:0:1$		$\overline{25}\,.\,\overline{34}$
$1:1:0$		$\overline{23}\,.\,\overline{45}$

The points or directions of this system, $\overline{ij}\,.\,\overline{kl}$, are invariant under the 15 distinct subgroups

$$G_8^{ij\,.\,kl} \sim \{ikjl\}_8, \qquad (i, j, k, l = 1 \,\ldots\, 5).$$

44. *Quintuple points.*

The points* $\quad -\lambda : 1 + \lambda : 1$ and $-\lambda' : 1 + \lambda' : 1$

are fixed under the transformation of period 5 corresponding to

$$(12543),$$

$*\lambda = \frac{1+\sqrt{5}}{2}, \quad \lambda' = \frac{1-\sqrt{5}}{2}.$

and are invariant under the subgroup

$$G_{10}^{12543} \sim \{12543\}_{10} \equiv [\{12543\}_{20}] \text{ pos.}$$

These points may be named respectively

$$12543 \text{ and } 14235,$$

with the understanding that the meaning of the notation is unchanged so long as the same *direct* or *reverse cyclic order* is maintained, thus

$$12543 = 25431 = 54321 = \ldots\ldots = 13452 = 34521 = \ldots\ldots$$
$$14235 = 42351 = 23514 = \ldots\ldots = 15324 = 53241 = \ldots\ldots$$

This system is as follows:

Quintuple Points.	Notation.
$-\lambda : 1+\lambda : 1$	12543
$-\lambda' : 1+\lambda' : 1$	14235
$1+\lambda : -\lambda : 1$	12453
$1+\lambda' : -\lambda' : 1$	14325
$\lambda : -\lambda' : 1$	15423
$\lambda' : -\lambda : 1$	12534
$1+\lambda : \lambda : 1$	15243
$1+\lambda' : \lambda' : 1$	14532
$-\lambda : \lambda' : 1$	15342
$-\lambda' : \lambda : 1$	14523
$\lambda : 1+\lambda : 1$	13524
$\lambda' : 1+\lambda' : 1$	12345

These points are fixed by twos under the 6 distinct conjugate subgroups; thus the two points

$$1ijkl \text{ and } 1kilj$$

are fixed under the subgroup

$$G_{10}^{1ijkl} \sim [\{1ijkl\}_{20}] \text{ pos.,} \qquad (i, j, k, l, = 2 \ldots\ldots 5),$$

45. There remain for consideration two systems consisting of imaginary fixed points which are not intersection points in the configuration Π.

One such system belongs to transformations of Period 4. Thus, the points*

$$1-i:-i:1 \text{ and } 1+i:i:1$$

are fixed under the transformation $\sim (2345)$, and are invariant under the subgroup

$$G_4^{2345} \sim \{2345\} \text{ cyc.}$$

These points may be named respectively

$$2345 \text{ and } 2543,$$

with the understanding that the meaning of the notation is unchanged so long as the same direct cyclic order is maintained, thus:

$$2345 = 3452 = 4523 = 5234,$$
$$2543 = 5432 = 4325 = 3254.$$

This system is as follows:

30 Complex Points.	Notation.
$2:1+i:1$	1234
$2:1-i:1$	1432
$1:1+i:2$	1243
$1:1-i:2$	1342
$-1:-i:1$	1324
$-1:+i:1$	1423
$1+i:2:1$	1235
$1-i:2:1$	1532
$1+i:1:2$	1253
$1-i:1:2$	1352
$-i:-1:1$	1325
$+i:-1:1$	1523
$1:2:1+i$	1245
$1:2:1-i$	1542

* $i=\sqrt{-1}$.

30 Complex Points.	Notation.
$2 : 1 : 1+i$	1254
$2 : 1 : 1-i$	1452
$+i : -i : 1$	1425
$-i : +i : 1$	1524
$1 : 1+i : 1-i$	1345
$1 : 1-i : 1+i$	1543
$1+i : 1 : 1-i$	1354
$1-i : 1 : 1+i$	1453
$1-i : 1+i : 1$	1435
$1+i : 1-i : 1$	1534
$1-i : -i : 1$	2345
$1+i : +i : 1$	2543
$-i : 1-i : 1$	2354
$+i : 1+i : 1$	2453
$1 : -i : 1-i$	2435
$1 : +i : 1+i$	2534

These points lie in pairs on the 15 conics and diagonals, namely, the two points

$$jklm \text{ and } jmlk$$

lie on

$$jl.km.$$

They also belong in pairs to the 15 distinct conjugate subgroups

$$G_4^{jklm} \sim \{jklm\} \text{ cyc.,} \qquad (j, k, l, m, = 1 \ldots . 5).$$

46. Finally, the points

$$-\omega : 0 : 1 \text{ and } -\omega^2 : 0 : 1$$

are fixed under transformation of Periods 3 and 6, and belong to the subgroup

$$G_6^{134} \sim \{134\} \text{ cyc. } \{25\}.$$

And the directions

$$\omega z_1 + z_2 = 0 \text{ and } \omega^2 z_1 + z_2 = 0$$

are invariant under

$$G_6^{245} \sim \{245\} \text{ cyc. } \{13\}.$$

These points and directions may be named respectively

$$134, \; 143, \; 245, \; 254,$$

the *direct cyclic order only being definitive.*

This system is as follows:

"Direction" Tangents in the	Pencils.	Notation.
$\omega z_1 + z_2 + \omega^2 z_3 = 0$	12	345
$\omega^2 z_1 + z_2 + \omega z_3 = 0$	12	354
$\omega z_1 + z_2 = 0$	13	245
$\omega^2 z_1 + z_2 = 0$	13	254
$z_2 + \omega z_3 = 0$	14	235
$z_2 + \omega^2 z_3 = 0$	14	253
$z_1 + \omega z_3 = 0$	15	234
$z_1 + \omega^2 z_3 = 0$	15	243
Complex Points on the	Sides.	
$1 : -\omega^2 : 0$	23	145
$1 : -\omega : 0$	23	154
$0 : -\omega^2 : 1$	24	135
$0 : -\omega : 1$	24	153
$-\omega : 0 : 1$	25	134
$-\omega^2 : 0 : 1$	25	143
$1 ; -\omega^2 : 1$	34	125
$1 : -\omega : 1$	34	152
$-\omega : 1 : 1$	35	124
$-\omega^2 : 1 : 1$	35	142
$-\omega : -\omega : 1$	45	123
$-\omega^2 : -\omega^2 : 1$	45	132

These points or directions lie in pairs on the 10 sides and pencils, and belong in pairs to the 10 distinct conjugate subgroups; thus the points ijk and ikj belong to

$$G_6^{ijk} = G_6^{ikj} \sim \{ijk\} \text{ cyc. } \{lm\},$$
$$(i, j, k, l, m, = 1 \ldots . 5).$$

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47. In the preceding grouptheoretic study, we have found loci of two types:

(a) The $6 + 4$ sides and pencils ij.

(b) The $3 + 12$ diagonals and conics $ij \,.\, kl$;

and real intersection points of three types:

(a) The 30 double points $\overline{ij} \,.\, kl$.

(b) The 15 quadruple points $\overline{ij} \,.\, \overline{kl}$.

(c) The 12 quintuple points $ijklm$, $[i, j, k, l, m, = 1 \ldots . 5]$.

48. These points are distributed in the following manner:

(1) On the line of type $ij \,.\, kl$ are two points,

$$\overline{ij} \,.\, kl \text{ and } \overline{kl} \,.\, ij .$$

On the line (or pencil) of type ij are 3 points or directions,

$$\overline{ij} \,.\, kl, \quad \overline{ij} \,.\, km, \quad \overline{ij} \,.\, lm .$$

Hence we enumerate

$$2 \,.\, 15 = 3 \,.\, 10 = 30 \text{ double points.}$$

(2) On the line $ij \,.\, kl$ are 2 points

$$\overline{ik} \,.\, \overline{jl} \text{ and } \overline{il} \,.\, \overline{jk},$$

and through such a point $\overline{ik} \,.\, \overline{jl}$ pass 2 curves

$$ij \,.\, kl \text{ and } il \,.\, kj .$$

On the line of type ij are 3 points,

$$\overline{ij} \,.\, \overline{kl}, \quad \overline{ij} \,.\, \overline{km}, \quad \overline{ij} \,.\, \overline{lm},$$

and through such a point $\overline{ij} \,.\, \overline{kl}$ pass 2 lines

$$ij \text{ and } kl,$$

thus giving

$$\frac{2 \,.\, 15}{2} = \frac{3 \,.\, 10}{2} = 15 \text{ quadruple points.}$$

(3) On a curve of type $ij \,.\, kl$ are 4 points,

$$miklj, \quad milkj, \quad mkijl, \quad mkjil,$$

and through such a point $mik\,lj$ pass 5 curves,

$$ij\,.\,kl, \quad mi\,.\,kj, \quad mk\,.\,lj, \quad ml\,.\,ik, \quad mj\,.\,il,$$

thus enumerating

$$\frac{4\,.\,15}{5} = 12 \text{ quintuple points.}$$

49. A line of type ij contains 6 intersection points, always in the cyclic order,

$$4, \ 2, \ 4, \ 2, \ 4, \ 2, \qquad\qquad [\text{see Fig. X}],$$

where the numbers indicate the multiplicity of the points, thus:

(a) On the finite side 25:

$$\overline{14}\,.\,\overline{25}, \ \overline{25}\,.\,34, \ \overline{13}\,.\,\overline{25}, \ \overline{25}\,.\,14, \ \overline{25}\,.\,\overline{34}, \ \overline{25}\,.\,13.$$

(b) On the line at infinity 23, starting at $\overline{23}\,.\,45$, the intersection with the diagonal $23\,.\,45$, and reading clockwise,

$$\overline{23}\,.\,45, \ \overline{15}\,.\,\overline{23}, \ \overline{23}\,.\,14, \ \overline{23}\,.\,\overline{45}, \ \overline{23}\,.\,15, \ \overline{14}\,.\,\overline{23}.$$

(c) In a finite pencil 13, starting at $\overline{13}\,.\,\overline{25}$, the intersection with the side 24, and reading counter-clockwise [see Fig. XI],

$$\overline{13}\,.\,\overline{25}, \ \overline{13}\,.\,24, \ \overline{13}\,.\,\overline{45}, \ \overline{13}\,.\,25.\,\overline{13}\,.\,\overline{24}, \ \overline{13}\,.\,45,$$

(d) In a pencil at infinity 14, starting at $\overline{14}\,.\,\overline{23}$, where the parabolas $12\,.\,34$ and $13\,.\,24$ are tangent to the line at infinity 23, and reading counter-clockwise,

$$\overline{14}\,.\,\overline{23}, \ \overline{14}\,.\,25, \ \overline{14}\,.\,\overline{35}, \ \overline{14}\,.\,23, \ \overline{14}\,.\,\overline{25}, \ \overline{14}\,.\,35.$$

50. A line of type $ij\,.\,kl$ has 8 intersection points always in the cyclic order,

$$2, \ 5, \ 4, \ 5, \ 2, \ 5, \ 4, \ 5,$$

the numbers indicating the multiplicity, thus:

(a) On a diagonal $23\,.\,45$:

$$\overline{23}\,.\,45, \ 12543, \ \overline{24}\,.\,\overline{35}, \ 14325, \ \overline{45}\,.\,23, \ 14235, \ \overline{25}\,.\,\overline{34}, \ 12453.$$

(b) On the hyperbola $15\,.\,23$, starting in the pencil 15 and reading continuously along the curve

$$\overline{15}\,.\,23, \ 14523, \ \overline{12}\,.\,\overline{35}, \ 15243, \ \overline{23}\,.\,15, \ 15342, \ \overline{13}\,.\,\overline{25}, \ 14532.$$

(c) On an equilateral hyperbola 15 . 34, starting in the pencil 14,

$\overline{14}.\overline{35}$, 12543, $\overline{15}$. 34, 12534, $\overline{13}$. 45, 14235, $\overline{34}$. 15, 15423.

(d) On a parabola 12 . 35, starting at infinity and reading counter-clockwise,

$\overline{15}.\overline{23}$, 12543, $\overline{35}$. 12, 12345, $\overline{13}.\overline{25}$, 14235, $\overline{12}$. 35, 13524.

It is to be noted here that $\overline{15}.\overline{23}$ is a direction in the pencil 15, whose tangent 23 is common to the two parabolas,

12 . 35 and 13 . 25,

and thus the pencil 15 is said to be cut by

23, 12 . 35 and 13 . 25,

forming the equivalent of a quadruple point. [Art. 40.]

Fundamental Region for G_{120}. Arts. 51–65.

51. If the fundamental region is known for any subgroup, that of the main group may be found by successively extending the subgroup and subdividing its region.

Let F_i represent the fundamental region for the subgroup G_i, then for

$$G_1 = \{1\}, \quad F_1 = \text{the whole plane.}$$

52. The extender

$$L \sim (23)(45)$$

leads to the subgroup

$$G_2 = \{L\} \sim \{23 . 45\},$$

with the relation $L_2 = 1$.

F_2 is then defined* by

$$z_3 \geqq 0,$$

$$z_1 + z_2 - z_3 \geqq 0. \qquad \text{[See Fig. VII.]}$$

53. Again, the extender

$$M \sim (45)$$

leads to the subgroup

$$G_4 = \{L, M\} \sim \{(23)(45)\}$$

---

* With the understanding that z_1, z_2, z_3 are all + above $z_2 = 0$ and to the right of $z_1 = 0$.

with the relations

$$L^2 = M^2 (LM)^2 = 1 .$$

The defining inequalities of F_4 are now

$$\begin{aligned} z_3 &\geqq 0 , \\ z_1 - z_2 &\geqq 0 , \\ z_1 + z_2 - z_3 &\geqq 0 . \end{aligned}$$

[See Fig. VIII.]

The generators L and M are edge-operators on the boundaries of F_4, and since the repetition of these operators can introduce no new lines, F_4 is at once a *simple region.*

54. The extender $K \sim (34)$ leads to the subgroup

$$G_{24}^{(1)} = \{K, L, M\} \sim \{2345\} \text{ all}$$

with the relations [see Fig. II],

$$K^2 = L^2 = M^2 = (LM)^2 = (MK)^3 = LK)^4 = 1 .$$

Since the index of G_4 under $G_{24}^{(1)}$ is 6, it follows that F_4 should be partitioned into 6 simple regions by the generators of $G_{24}^{(1)}$. In fact, K, L, M are edge-operators on the lines

34, 23.45 and 45.

It is easy to show that the repetition of these transformations gives no lines entering the region F_4 except

23, 34, 35 and 23.45,

and these produce the six subdivisions defined as follows:

$$\begin{array}{lrrr}
\text{(I).} & z_1 + z_2 - z_3 \geqq 0, & z_1 - z_2 \geqq 0, & z_1 - z_3 \leqq 0 . \\
\text{(II).} & z_1 - z_3 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \leqq 0 . \\
\text{(III).} & z_1 - z_2 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \leqq 0 . \\
\text{(IV).} & z_3 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \geqq 0 . \\
\text{(V).} & z_2 \geqq 0, & z_2 - z_3 \geqq 0, & z_1 - z_2 - z_3 \geqq 0 . \\
\text{(VI).} & z_3 \geqq 0, & z_2 \leqq 0, & z_1 + z_2 - z_3 \geqq 0 .
\end{array}$$

56. By use of these inequalities and the three generators K, L, M, it may be readily shown:

(a) Each of these six regions is simple; that is, it is not subdivided by any line of Π_1.

(b) Any point in the plane *outside* one of these regions, say (1), has *one* and *only one* conjugate point within that region.

(c) Any point *within* one of these regions has no other conjugate point within it.

We may then name region (1) F_{24}, in agreement with Art. 8, (6), and the others in order will be

$$K,\ KM,\ KLM,\ KL,\ KLK. \qquad \text{See Fig. IX.}$$

Since each boundary of F_{24} is a fixed axis of reflection for one of the generators, it follows that the boundaries, including the corner points, count as a part of the region F_{24}, and hence of every conjugate region in Π_1.

57. Applying the six extenders to G_4, we get the rectangular table for $G_{24}^{(1)}$:

1	L	M	LM
K	LK	LM	LMK
KM	LKM	MKM	$LMKM$
KL	LKL	MKL	$LMKL$
KLK	$LKLK$	$MKLK$	$LMKLK$
KLM	$LKLM$	$MKLM$	$LMKLM$

These are the operators of $G_{24}^{(1)}$ as shown in Fig. II.

58. Finally, the extender

$$T \sim (15)(34)$$

leads to the main group

$$G_{120} \sim \{12345\} \text{ all.}$$

It is to be shown that F_{24} is divided into *five simple regions* by the lines and conics of the configuration Π, Fig. X.

To show what curves enter F_{24}.

(a) No curve of Π_1 can cut F_{24}, since this is a fundamental region for $G_{24}^{(1)}$.

(b) Hence, the only possibilities of partition within F_{24} are by the 12 conics.

Of these, it has been shown [§4] that none pass through the vertex $\overline{25}\,.\,\overline{34}$, none through the vertex $\overline{45}\,.\,23$, 9 through the vertex 12, one through point $\overline{34}\,.\,15$, on the boundary 34, none through any point on the boundary 45, and 4 through the point 14235 on the boundary 23 . 45.

There are thus only three possible points of entrance into F_{24},

$$12,\ \overline{34}\,.\,15 \text{ and } 14235\,.$$

It remains to show which of the conics through these points actually cut the boundary and enter the region. For this purpose it is convenient to use non-homogeneous coordinates, putting

$$\frac{z_1}{z_3}=\rho,\quad \frac{z_2}{z_3}=\sigma\,.$$

59. The interior of F_{24} is then defined by

$$\rho-1<0, \tag{1}$$

$$\rho-\sigma>0, \tag{2}$$

$$\rho+\sigma-1>0\,. \tag{3}$$

From (1) and (3),

$$\sigma>0\,. \tag{4}$$

From (2) and (3),

$$\rho>\tfrac{1}{2}\,. \tag{5}$$

Consider any one of the conics through 14235, say 13 . 24,

$$\sigma^2-2\sigma+\rho=0, \tag{6}$$

from which

$$\rho=2\sigma-\sigma^2. \tag{7}$$

Substitute (7) in (1),

$$2\sigma-\sigma^2-1<0\,.$$

Hence,

$$(\sigma-1)^2>0\,. \tag{8}$$

Substitute (7) in (2).

$$2\sigma-\sigma^2-\sigma>0,$$

or

$$\sigma-\sigma^2>0\,.$$

Hence,

$$\sigma>0\,. \tag{9}$$

which agrees with (4).

Substitute (7) in (3),

$$2\sigma-\sigma^2+\sigma-1>0,$$

or

$$\sigma-(\sigma-1)^2>0\,.$$

Hence,

$$\sigma>0\,. \tag{10}$$

which agrees with (4).

Since, by (8), (9) and (10), the defining inequalities of F_{24} are consistent with (7), therefore, the conic 13.24 enters the region F_{24}.

In like manner the other conics

$$12.35, \quad 14.25 \text{ and } 15.24$$

may be shown to *cut* the boundary at 14235 and enter the region F_{24}.

60. Of these four conics, 15.34 passes through the point $\overline{34}.15$, and hence must *cut* the boundary 34 at that point. The other three pass through the vertex 12, and hence must pass out of F_{24} at that point.

Moreover, *none of the remaining* 9 *conics through* 12 *can enter the region* F_{24}, since the only points of exit, $\overline{34}.15$ and 14235, have already their maximum number.

Hence, precisely 4 conics pass through the region F_{24}, and, therefore, *the partition into at least* 5 *parts is established.*

61. *To show that each of these divisions is a simple region,* we name and define them in order as follows, starting at the vertex $\overline{25}.\overline{34}$ and passing along the boundary 23.45 toward the vertex $\overline{45}.23$.

$$\begin{array}{lllll}
1 & ; & \rho - 1 \leqq 0, & \rho + \sigma - 1 \geqq 0, & \rho\sigma - \rho + 1 \leqq 0. \\
T & ; & \rho - 1 \leqq 0, & \rho\sigma - \rho + \sigma \geqq 0, & \sigma^2 - 2\sigma + \rho \geqq 0. \\
TL & ; & \rho^2 - \sigma \geqq 0, & \sigma^2 - 2\sigma + \rho \leqq 0. & \\
TLT & ; & \rho^2 - \sigma \leqq 0, & \rho\sigma - 2\rho + 1 \geqq 0. & \\
TLTL & ; & \rho - \sigma \geqq 0, & \rho + \sigma - 1 \geqq 0, & \rho\sigma - 2\rho + 1 \geqq 0.
\end{array}$$

The only possibility of subdivision in any one of these regions is by one of the 4 conics (just shown to enter F_{24}), which is *not* a *boundary* of the region in question. For instance, the region 1, whose interior is defined by

$$\rho - 1 < 0, \tag{1}$$
$$\rho + \sigma - 1 > 0, \tag{2}$$
$$\rho\sigma - \rho + \sigma > 0, \tag{3}$$

could be subdivided only by

$$12.35; \quad \rho^2 - \sigma = 0, \tag{4}$$

or

$$13.24; \quad \sigma^2 - 2\sigma + \rho = 0, \tag{5}$$

or
$$14.25;\quad \rho\sigma - 2\rho + 1 = 0. \tag{6}$$
Substitute the value of σ from (4) in (2),
$$\rho^2 + \rho - 1 > 0. \tag{7}$$
Substitute the same in (3) and find
$$\rho^2 + \rho - 1 < 0. \tag{8}$$

As (7) and (8) are contradictory, 12.35 does not enter region 1.

Substitute the value of ρ from (5) in (3),
$$\sigma^2 - 3\sigma - 1 > 0. \tag{9}$$
Also in (2),
$$-\sigma^2 + 3\sigma - 1 > 0. \tag{10}$$
Add (9) and (10),
$$-2 > 0.$$
As this is impossible, 12.34 has no points with region 1.

Finally, substitute the value of ρ from (6) in (3) and find
$$\sigma^2 - 3\sigma + 1 > 0. \tag{11}$$
Also in (2),
$$-\sigma^2 + 3\sigma - 2 > 0. \tag{12}$$
Add (11) and (12),
$$-1 > 0.$$
Hence, 14.25 does not enter the region 1. Therefore, 1 is a simple region, and in the same manner each of the other four regions may be proved simple.

The names already given to these four regions correspond to the transformations by which they are respectively derived from the region 1.

62. These transformations
$$1,\ T,\ TL,\ TLT,\ TLTL,$$
are the proper extenders with which to form the rectangular table for G_{120} from the operators of $G_{24}^{(1)}$, for

(a) They are all different from those of $G_{24}^{(1)}$, since no two points in F_{24} can be conjugate under $G_{24}^{(1)}$.

(b) They are distinct from one another, since the supposition of equality between any two of them leads to a contradiction.

Such a table would contain initially the four generators
$$K,\ L,\ M,\ T,$$
from which, however, M may be eliminated by the relation
$$M = TKLKTLTK, \tag{1}$$

where T, L and K are *edge-operators on the boundaries of region* 1; *that is, each generator has for its axis of reflection one of the boundary curves which it leaves fixed by points.*

The table is not given in full, but the thirty subdivisions of F_4 are marked in Fig. XI, and the others (some of which are indicated), may be read at once by applying to these the transformations of

$$G_4 = \{L, M\} \sim \{(23)(45)\},$$

together with the relation (1).

63. Since F_{24} is shown to be partitioned into 5 simple regions, therefore, so is each of the 24 divisions of Π_1.

Hence the configuration Π contains precisely 120 regions and the transformations of G_{120} throw region 1 to these 120 regions respectively, each bearing uniquely the name of the transformation by which it is derived from region 1.

Therefore, the conditions are fulfilled for a fundamental region

$$1 = F_{120},$$

in which the boundaries and vertices count as a part of the region.

(a) No two points of F_{120} are conjugate under any transformation of G_{120}, since every such operator throws each point of F_{120} to a *conjugate point in the new region whose name is the transformation in question.*

(b) Any point in the plane has *at least one* conjugate point in F_{120}; for every point belongs to one of the 120 regions, and hence is thrown to a point in F_{120} by the inverse of the transformation naming that region.

(c) No point in the plane can be conjugate to *two points* of F_{120}, since these two points would then be conjugate to each other, contrary to (a).

64. F_{120} is a triangle, two of whose sides are of type $ij.kl$, and the third of type ij, while the vertices are double, quadruple, and quintuple points respectively, thus including all types of real points and lines in the configuration Π.

Hence, each of the 120 regions possesses the same characteristics. The *apparently* two-sided figures have a *pencil* as the third side of type $1i$, and *directions* in that pencil as the *double* and *quadruple* vertices. [See Art. 40.]

Thus, region TL has the boundaries

$$13.24,\ 12.35 \text{ and the pencil } 12$$

and it has the vertices

$$14235,\ \overline{12}.\overline{34} \text{ and } \overline{12}.35.$$

65. Certain generational relations may now be read from the combination of transformations about the edges and vertices of the fundamental region.

(1) Since K, L and T are reflections on the three boundaries of F_{120}, then

$$K^2 = L^2 = T^2 = 1.$$

(2) KT is a rotation about a double point in either direction through 180°.

(3) KL is a clockwise rotation about a quadruple point through 90°.

(4) LT is a clockwise rotation about a quintuple point through 72°.

Hence, $$(KT)^2 = (KL)^4 = (LT)^5 = 1.$$

The abstract generational conditions also require other relations in addition to the above.*

THE UNIVERSITY OF CHICAGO, *July* 13, 1900.

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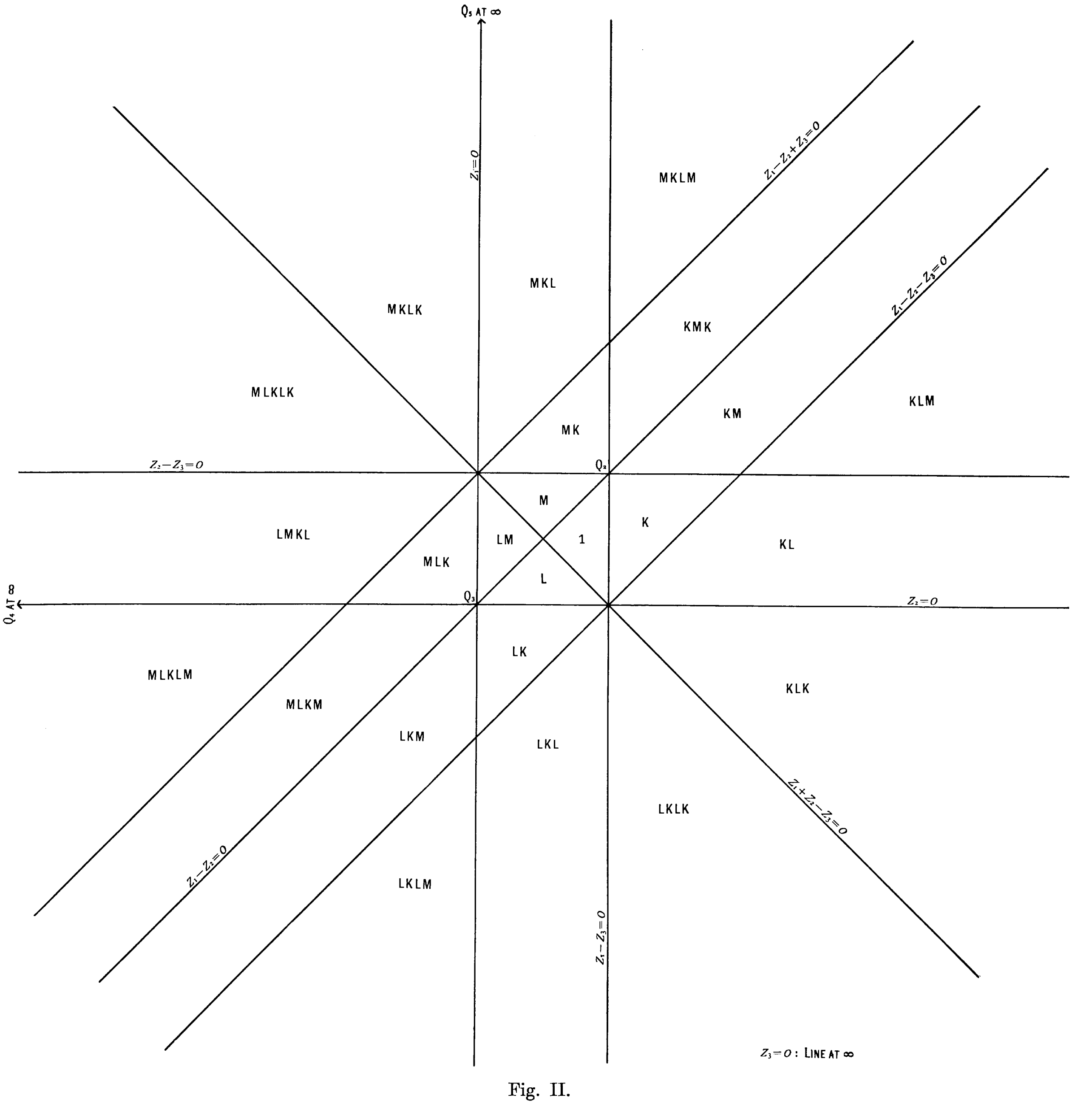
* E. H. Moore, "Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holoedrically Isomorphic with the Symmetric and Alternating Substitution Groups on k Letters" (Proceedings of the London Mathematical Society), vol. XXVIII, No. 597, pp. 357–366. Also, "Concerning Klein's Group of $(n+1)!$ n-ary Collineations" (American Journal of Mathematics, vol. XXII, pp. 341–342, §10, 1900).



Q_5 AT ∞
$Z_1=0$
MKLM
$Z_1-Z_2+Z_3=0$
MKL
MKLK
KMK
$Z_1-Z_2-Z_3=0$
MLKLK
KM
KLM
MK
$Z_2-Z_3=0$
Q_2
M
LMKL
K
LM
1
KL
MLK
L
Q_4 AT ∞
Q_3
$Z_2=0$
LK
MLKLM
KLK
MLKM
LKM
LKL
$Z_1+Z_2-Z_3=0$
LKLK
$Z_1-Z_2=0$
LKLM
$Z_1-Z_3=0$
$Z_3=0$: LINE AT ∞

Fig. II.

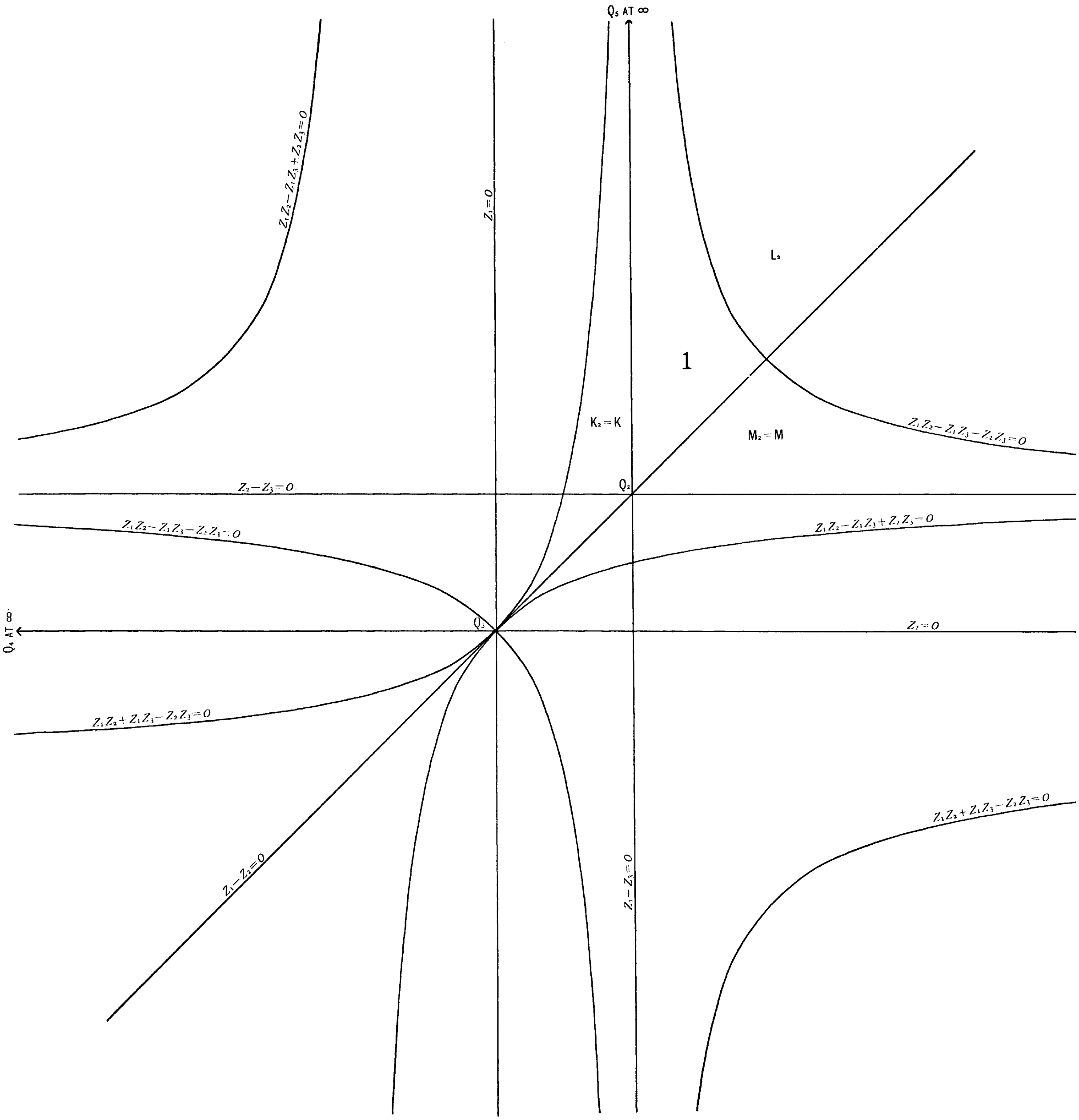
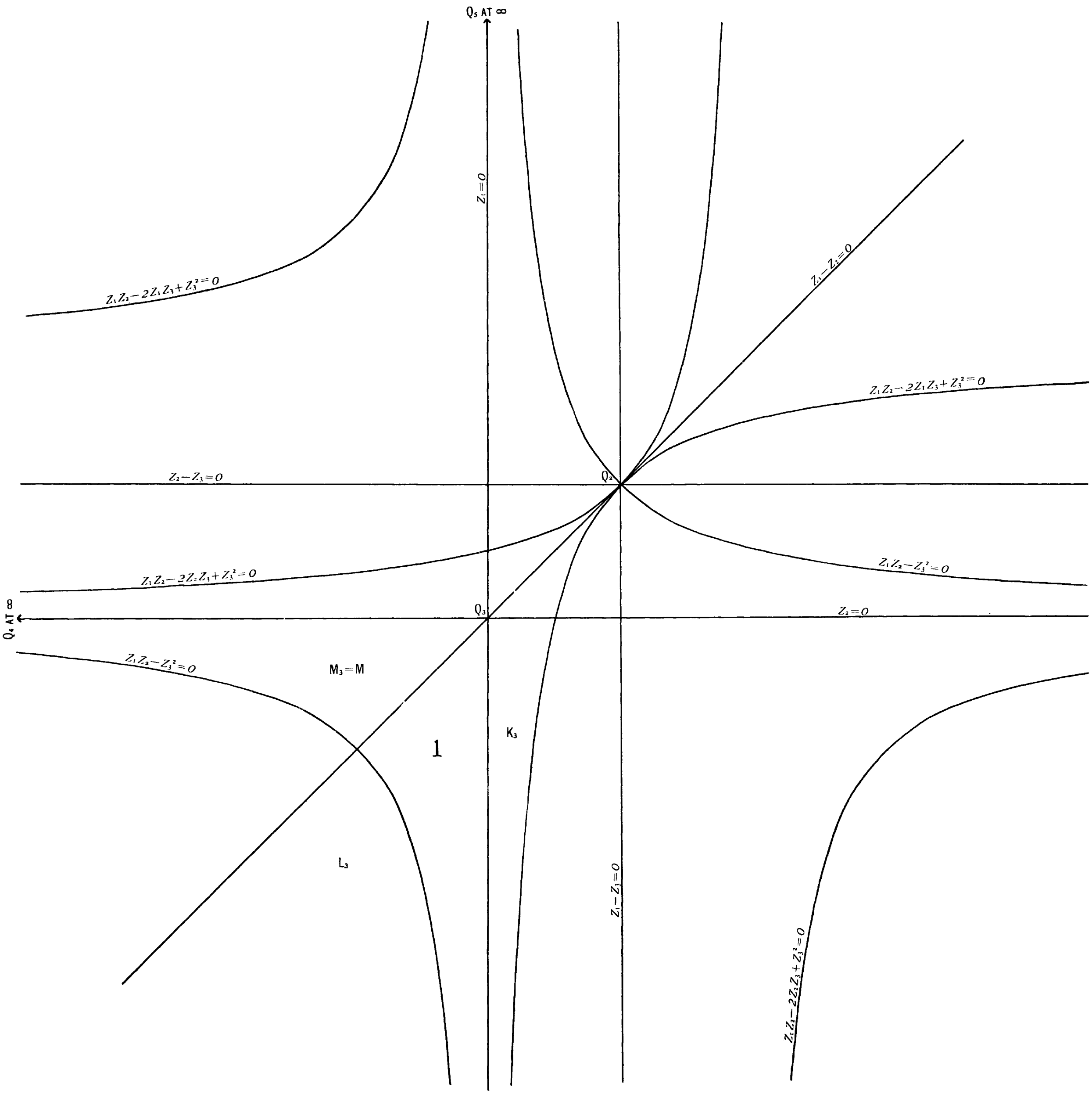
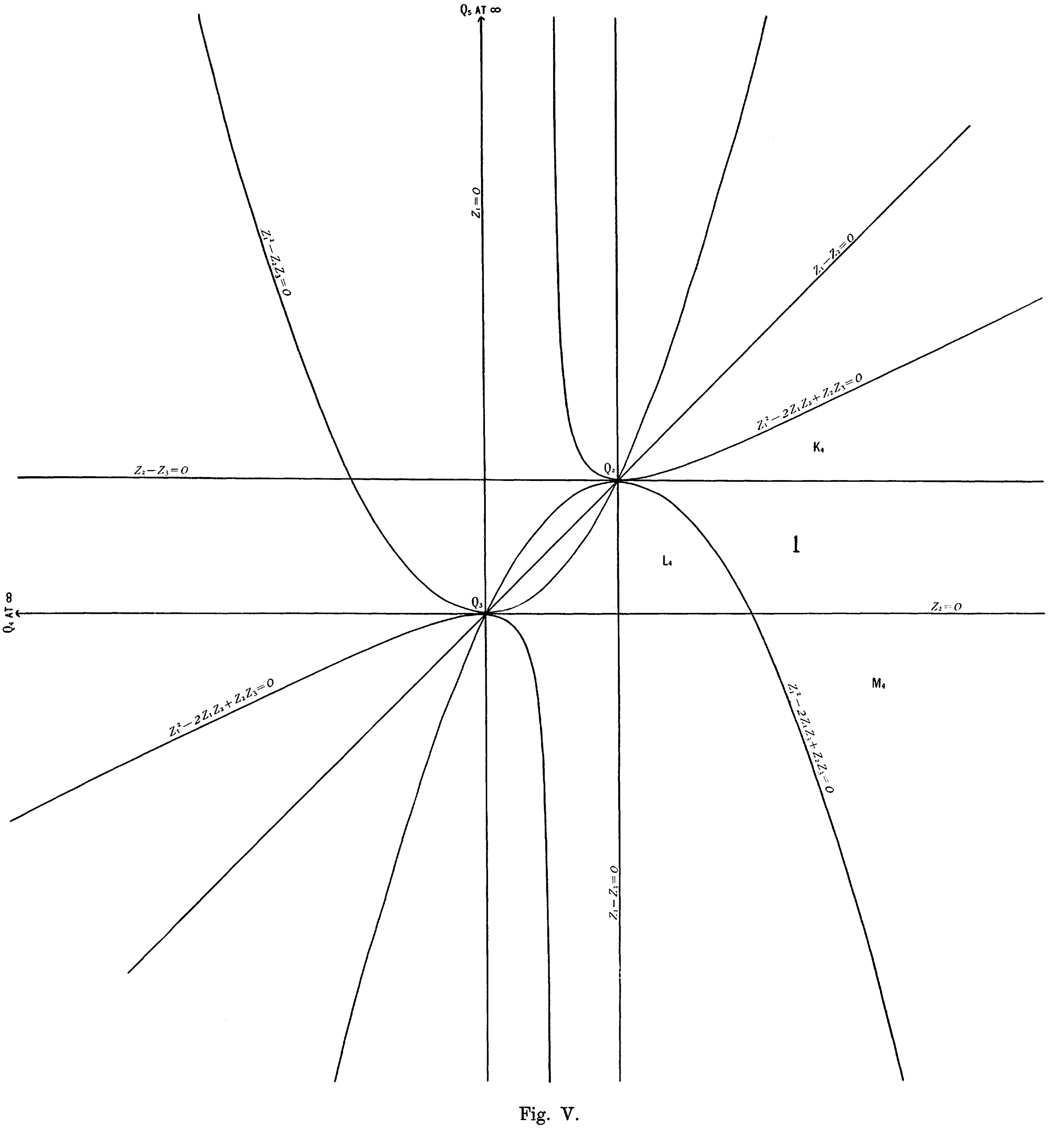

Q₅ AT ∞
Q₄ AT ∞
$Z_1Z_2-Z_1Z_3+Z_2Z_3=0$
$Z_1=0$
L_2
1
$K_2=K$
$M_2=M$
$Z_1Z_2-Z_1Z_3-Z_2Z_3=0$
$Z_2-Z_3=0$
Q_2
$Z_1Z_2-Z_1Z_3-Z_2Z_1=0$
$Z_1Z_2-Z_1Z_3+Z_2Z_3=0$
Q_3
$Z_2=0$
$Z_1Z_2+Z_1Z_3-Z_2Z_3=0$
$Z_1Z_2+Z_1Z_3-Z_2Z_3=0$
$Z_1-Z_3=0$
$Z_1-Z_3=0$

Fig. III.



Q_5 AT ∞
$Z_1=0$
$Z_1Z_2-2Z_1Z_3+Z_3^2=0$
$Z_1-Z_3=0$
$Z_1Z_2-2Z_1Z_3+Z_3^2=0$
$Z_2-Z_3=0$
Q_2
$Z_1Z_2-2Z_2Z_3+Z_3^2=0$
$Z_1Z_2-Z_3^2=0$
Q_4 AT ∞
Q_3
$Z_2=0$
$Z_1Z_2-Z_3^2=0$
$M_3=M$
K_3
1
L_3
$Z_1-Z_3=0$
$Z_1Z_2-2Z_2Z_3+Z_3^2=0$

Fig. IV.



Q_5 AT ∞
Q_4 AT ∞
$Z_1=0$
$Z_1^2-Z_2Z_3=0$
$Z_1-Z_3=0$
$Z_1^2-2Z_1Z_3+Z_2Z_3=0$
K_4
Q_2
$Z_2-Z_3=0$
1
L_4
Q_3
$Z_2=0$
M_4
$Z_1^2-2Z_1Z_2+Z_2Z_3=0$
$Z_1^2-2Z_1Z_3+Z_2Z_3=0$
$Z_1-Z_3=0$

Fig. V.

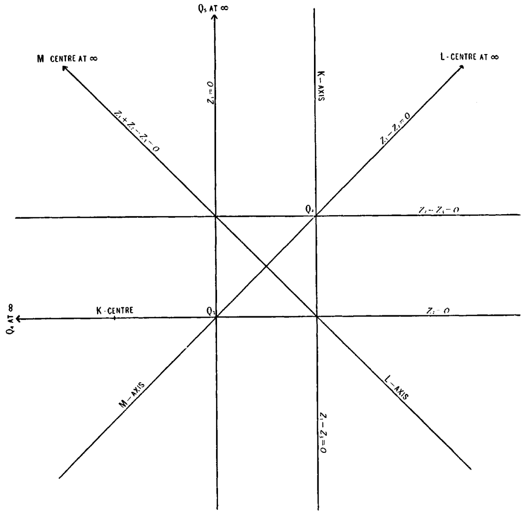


Fig. I.

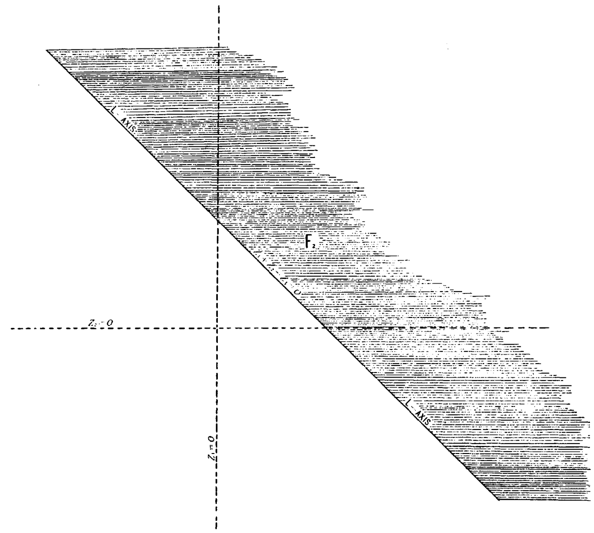


Fig. VII.

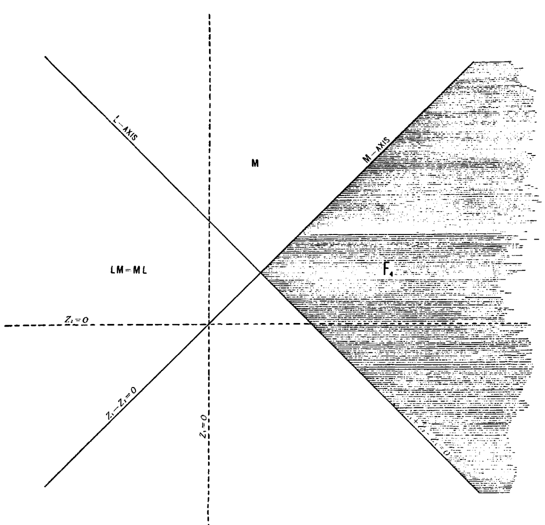


Fig. VIII.

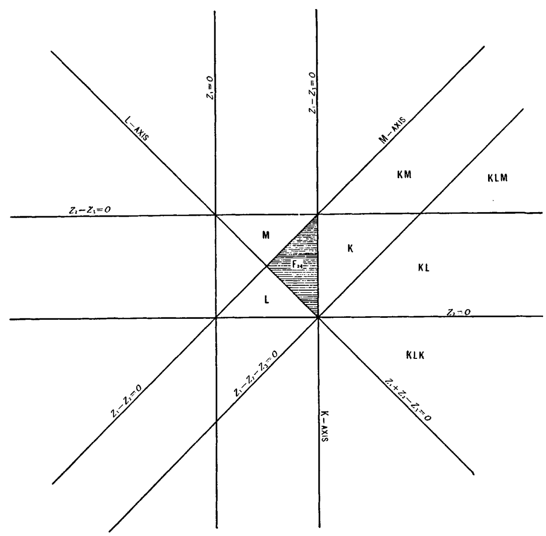
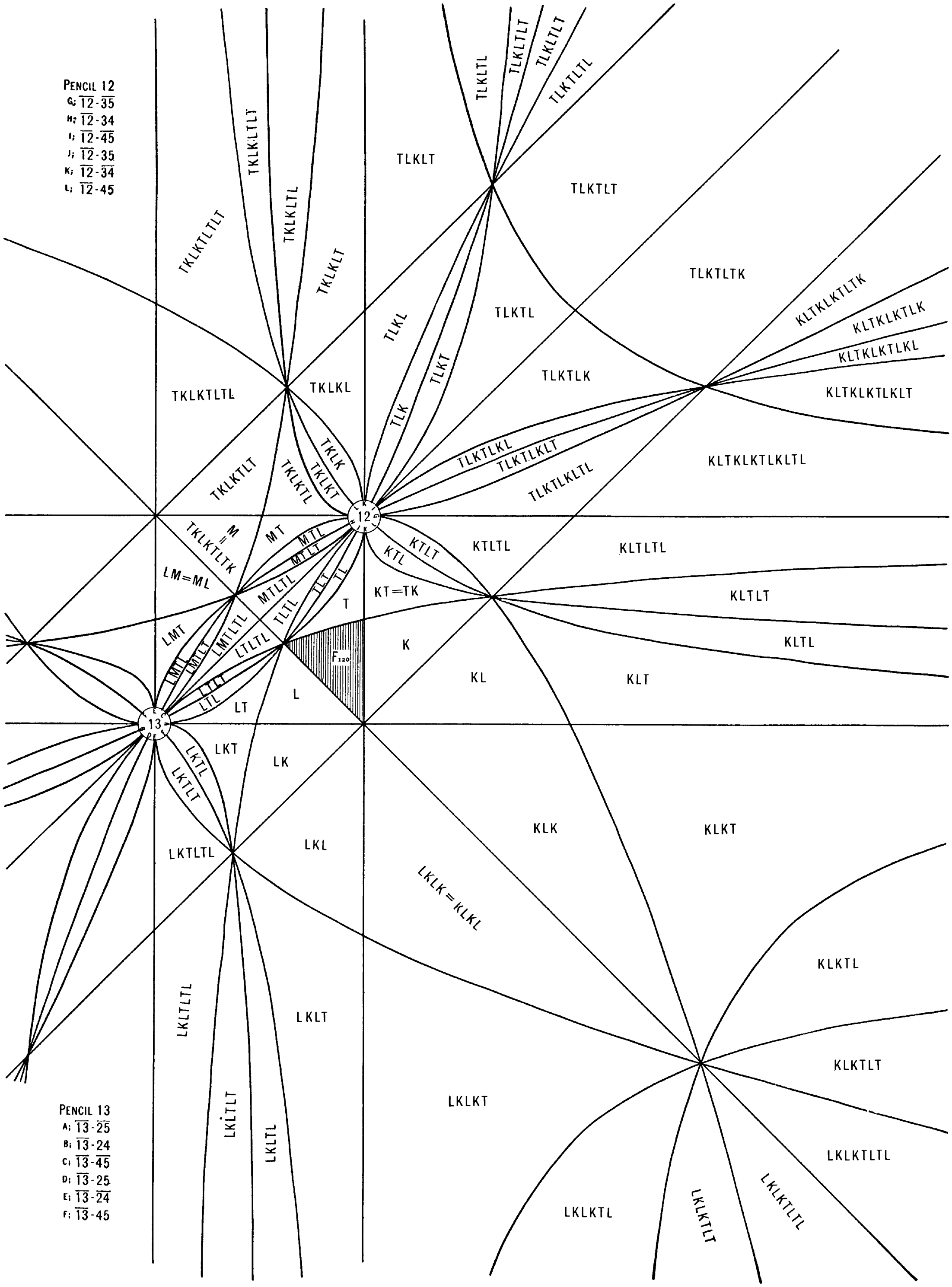


Fig. IX.


PENCIL 12
G; $\overline{12}$-$\overline{35}$
H; $\overline{12}$-34
I; $\overline{12}$-$\overline{45}$
J; $\overline{12}$-35
K; $\overline{12}$-$\overline{34}$
L; $\overline{12}$-45
PENCIL 13
A; $\overline{13}$-$\overline{25}$
B; $\overline{13}$-24
C; $\overline{13}$-$\overline{45}$
D; $\overline{13}$-25
E; $\overline{13}$-$\overline{24}$
F; $\overline{13}$-45
12
13
TLKLTL
TLKLTLT
TLKLTLT
TLKTLTL
TLKLT
TLKTLT
TKLKLTLT
TKLKLTL
TKLKTLTLT
TKLKLT
TLKTLTK
TLKTL
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KLTKLKTLTK
KLTKLKTLK
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TL
TLT
TLTL
LTLTL
LTL
LT
L
F_{120}
LM=ML
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KLKT
KLKTL
KLKTLT
LKLKTLTL
LKLKTLT
LKLKTL
LKLKT
LKLT
LKLTLTL
LKLTLT
LKLTL

Fig. XI.

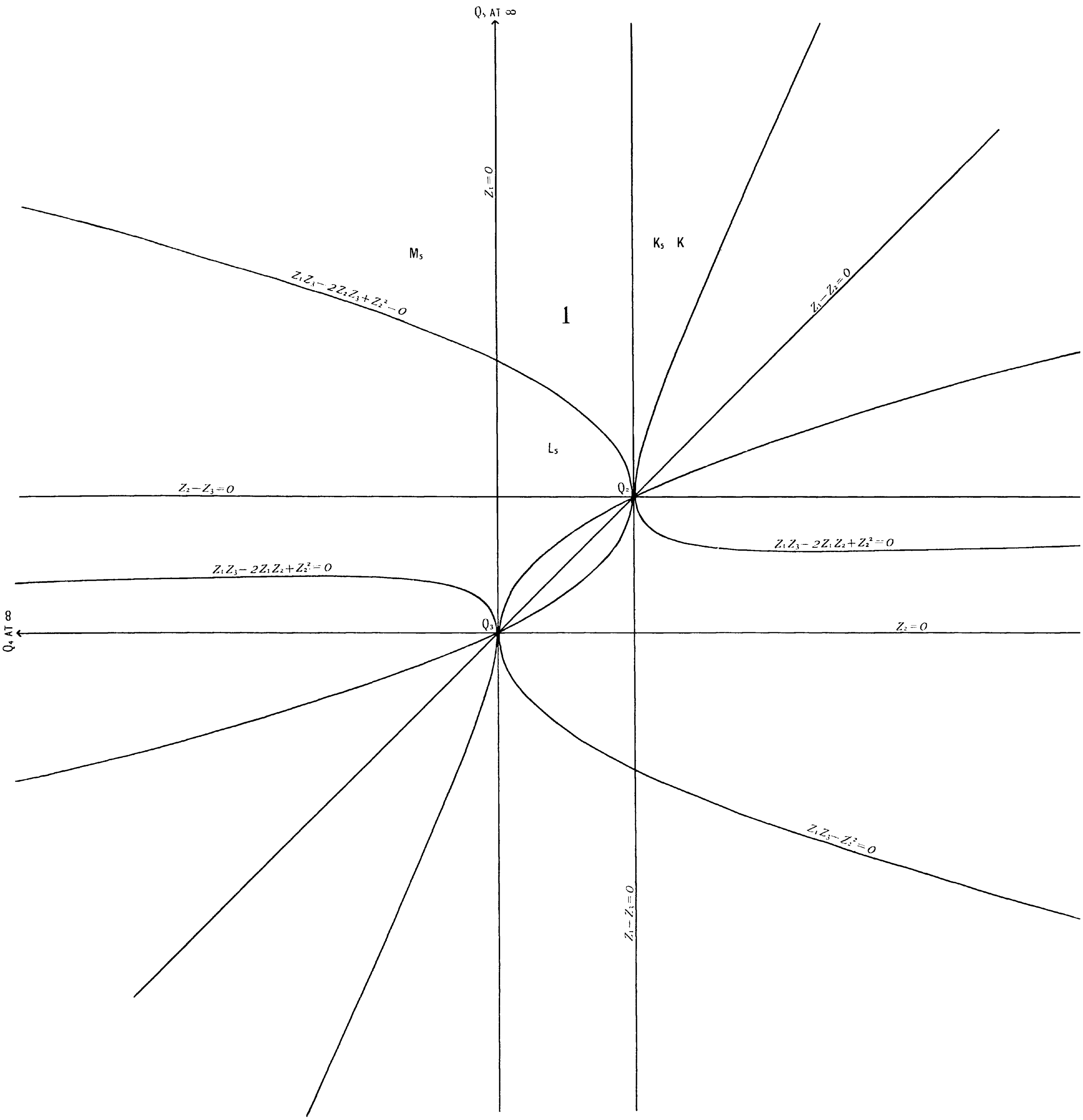
Q₅ AT ∞
Q₄ AT ∞
$Z_1=0$
M_5
K_5 K
1
L_5
$Z_1Z_3-2Z_2Z_3+Z_2^2=0$
$Z_1-Z_2=0$
$Z_2-Z_3=0$
Q_2
$Z_1Z_3-2Z_1Z_2+Z_2^2=0$
$Z_1Z_3-2Z_1Z_2+Z_2^2=0$
Q_3
$Z_2=0$
$Z_1Z_3-Z_2^2=0$
$Z_1-Z_3=0$

Fig. VI.

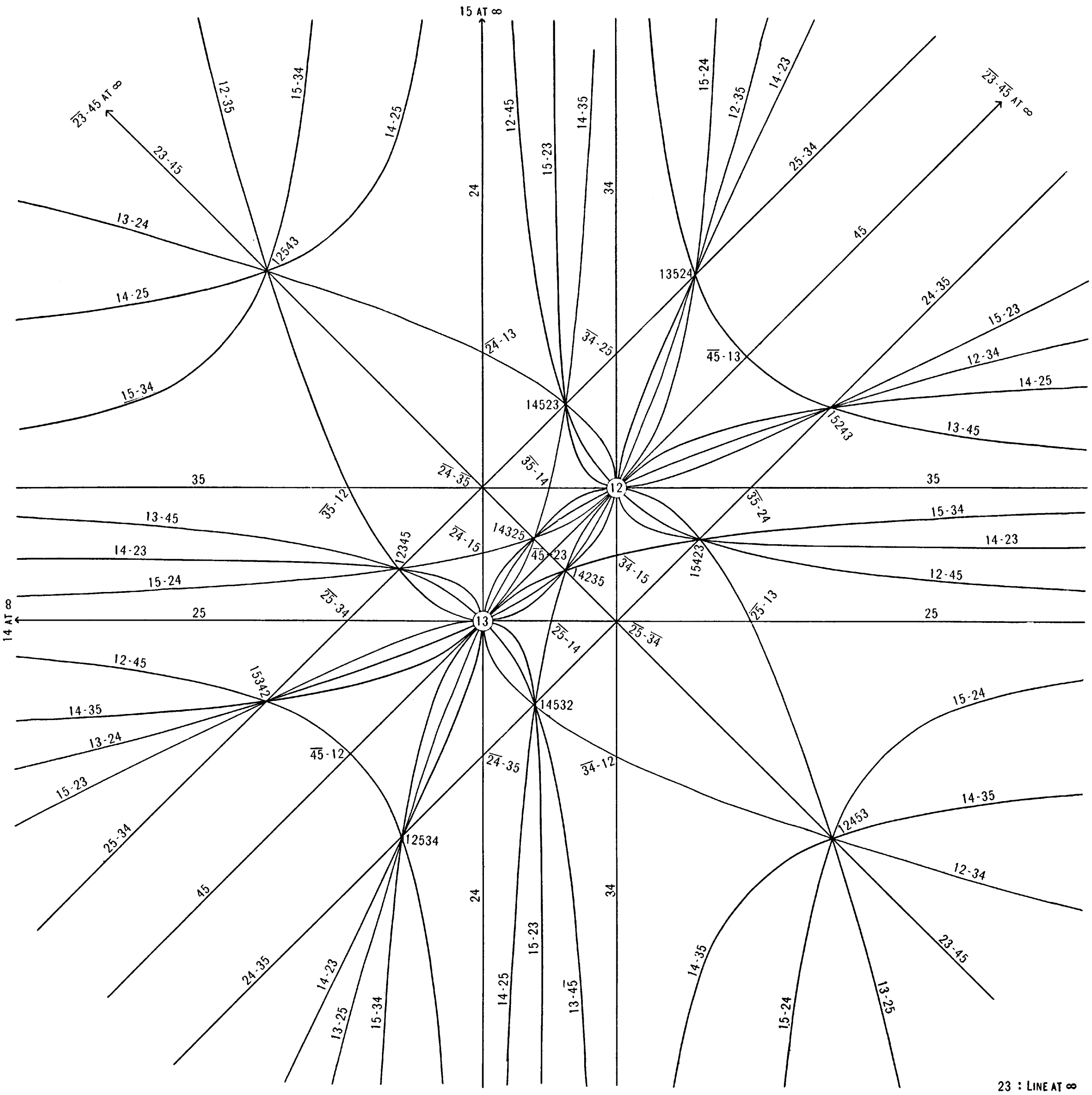
15 AT ∞
$\overline{23}$·45 AT ∞
$\overline{23}$·$\overline{45}$ AT ∞
12·35
15·34
14·25
23·45
13·24
12543
14·25
15·34
12·45
15·23
14·35
24
34
15·24
12·35
14·23
25·34
45
13524
24·35
15·23
12·34
14·25
13·45
15243
$\overline{24}$·13
$\overline{34}$·25
$\overline{45}$·13
14523
35
$\overline{24}$·$\overline{35}$
$\overline{35}$·14
12
35
$\overline{35}$·12
$\overline{35}$·24
13·45
14·23
15·24
12345
$\overline{24}$·15
14325
$\overline{45}$·23
14235
$\overline{34}$·15
15423
15·34
14·23
12·45
14 AT ∞
25
$\overline{25}$·34
13
$\overline{25}$·13
25
$\overline{25}$·14
$\overline{25}$·$\overline{34}$
12·45
15342
14·35
13·24
15·23
$\overline{45}$·12
14532
$\overline{24}$·35
$\overline{34}$·12
15·24
14·35
12453
25·34
12534
45
24
34
12·34
23·45
24·35
14·23
13·25
15·34
14·25
15·23
13·$\overline{45}$
14·35
15·24
13·25
23 : LINE AT ∞

Fig. X.